Strong Goldbach's conjecture proof Using a Structural Approach to numbers

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I. Abstract

This paper provides an elementary proof of Strong Goldbach's conjecture stating that *every even natural number greater than two can be written as the sum of two primes*. The conjecture addresses one of the oldest and most fundamental problems in number theory, which is understanding the distribution and properties of prime numbers.

The proof employs a structural approach to numbers. On one hand, it builds upon the understanding of the structure of primes, specifically focusing on their $6\mathrm{k} \pm 1$ patterns ($primes \geq 5$) as discussed in the opening section of the document, and on the other hand, it explores the underlying structure of even numbers, classifying them into three distinct categories based on their relationship to multiples of 3. Additionally, the proof employs a sieve logic encapsulated within an ad-hoc notation introduced in this paper (ad-hoc sieve notation), to provide the description for prime numbers into which an even number, according to its structure, can be decomposed. Finally, the proof covers formally the even numbers greater or equal to 10. The even numbers 4,6 and 8 are cases accepted as outliers in this approach since they involve the particular primes 2 and 3 that are not fitting any of $6\mathrm{k} \pm 1$ prime patterns (as: $4 = 2 + 2, 6 = 3 + 3$ and $8 = 3 + 5$).

At the conclusion of this paper, several practical and direct implications of this approach are discussed. These implications involve the identification of twin primes, new primality testing, identifying sequences of primes, and the articulation with the Goldbach's Comet.

Keywords: *Prime numbers, Primes, Strong Goldbach's conjecture, Twin primes, Primality test, Sexy primes, Series of primes, Goldbach's comet*

Mathematics Subject Classification – MSC2020: *Number theory – Primes: 11A41*

II. Introduction

The Strong Goldbach's Conjecture, also known as the Binary Goldbach's Conjecture or the 2-primes problem, is a famous unsolved problem that has intrigued mathematicians and enthusiasts for centuries. It was first proposed by the Prussian mathematician Christian Goldbach in a letter to Leonhard Euler on June 7, 1742 ([1]- Tattersall, 1999). The conjecture states that *every even natural number greater than 2 can be expressed as the sum of two prime numbers*[1]. Another variant of it called the Weak Goldbach's Conjecture, also known as the Odd Goldbach Conjecture, the Ternary Goldbach Problem, or the 3-Primes Problem, posits that *every odd number greater than 5 can be represented as the sum of three prime numbers*. The term "weak" is used because if Goldbach's strong conjecture is proven, this weaker version would also hold true ([2]- Wikipédia, n.d.). The Weak variant has gained widespread acceptance due to a publication by Harald Helfgott Anderson in 2013 ([3]- Helfgott., 2013). However, the Strong Goldbach Conjecture remains unsolved to this day. No counter-example has been found to date and the majority of mathematicians believe that the Strong conjecture is true. Significant progress has been denoted by strides in theoretical comprehension ([12]- Simone, 2018), with computational verification reaching up to 9×10^{18} ([4]- Daniel, 2023).

[1] Goldbach's considered number 1 as a prime is no longer in practice

III. Notations and conventions used

1. All discussed primes are ≥ 5: the targeted primes are the ones fitting the $6k \pm 1$ (all primes except 2 and 3)
2. $k, k_p, k_q, x, x', y, y', n, a, A, N$ are all natural numbers > 0 (unless explicit indication is mentioned)
3. $B \geq 0$ is a natural number, that can be odd or even
4. $N \geq 10$ is an even natural number and $A \geq 5$ natural number such that $N = 2A$
5. The letters p and q are used for primes that surround the number A with: $p \leq A \leq q$
6. $a \equiv b\ [n]$ notation is used for stating a is congruent to b modulo n.
7. The use of $a \not\equiv b\ [n]$ notation only in the ad-hoc sieve notation concept that is introduced in this paper. There are illustrations and examples in preliminary proposition section. The notation used for simplification purposes
8. There will be adherence to the designated roles of the letters of a, A, B, N, p, q letters as initially introduced to facilitate cross-referencing throughout the document sections
9. In the propositions and elsewhere, all the quantities $\left(\frac{a \pm B}{2} \pm n\right)$ are > 0

IV. Introduction of prime generators coefficients

M.KRAFT has introduced in 1798 in [*Essai sur les nombres premier par M. KRAFFT*] ([5]- KRAFFT, 1798) many properties related to structure of primes. In this section are presented some results we'll be directly using. the detailed proofs are deliberately not recalled.

Beside the numbers 2 and 3, all prime numbers come in one of the two patterns: 6k-1 or 6k+1 - (examples: 5, 11, 17 or 7, 13, 19), but not all the numbers having the form of $6k - 1$ or $6k + 1$ are primes – (ex.: 35,65,77 // 25,49,55). ([5]- KRAFFT, 1798)

The $6k - 1$ like non-prime numbers (called composite numbers) are having coefficients k that are belonging to a specific set of natural numbers that can be written under the form of: $6xy + x - y$ (where x, y ≥1 are natural numbers). ([5]- KRAFFT, 1798)

We'll refer to the set of these coefficients k as SET1:

$$\text{SET1} = \{n > 0 \text{ natural numbers, such that } \exists x, y > 0 \text{ that satify: } n = 6xy + x - y\}$$

[Justification: $6k - 1 = 6(6xy + x - y) - 1 = 36xy + 6x - 6y - 1 = (6x - 1)(6y + 1)$ ~ composite (6k-1) like]

The $6k + 1$ like non-prime numbers (or composite numbers) are having coefficients k that are belonging to another specific set of natural numbers that can be written either like $(6xy + x + y)$ or like $(6xy - x - y)$ where x, y ≥1 are natural numbers. ([5]- KRAFFT, 1798)

We'll refer to the set of these coefficients k as SET2:

$$\text{SET2} = \{n > 0 \text{ natural numbers, such that } \exists x, y > 0 \text{ that satify: } n = (6xy + x + y) \text{ or } n = (6xy - x - y)\}$$

[Justification:

$6k+1 = 6(6xy + x + y) + 1 = 36xy + 6x + 6y + 1 = (6x + 1)(6y + 1)$ ~ composite (6k+1) like

$6k' + 1 = 6(6xy - x - y) + 1 = 36xy - 6x - 6y + 1 = (6x - 1)(6y - 1)$ ~ composite (6k+1) like]

Primes number (in both shapes of $6k \pm 1$) are having k-coefficients that are not belonging to any of these specific sets (SET1 and SET2 described above). We'll refer to these k-coefficients as Primes Generators**.**

In other words: for a given coefficient k, to be a **prime generator**, there should be no natural numbers x, y > 0 such that: $k = 6xy + x - y$ or $k = 6xy + x + y$ or $k = 6xy - x - y$ ([5]- KRAFFT, 1798)

In other terms:

- if $k \notin \text{SET1} \Leftrightarrow$ k is a prime generator for $6k - 1$ pattern
- if $k \notin \text{SET2} \Leftrightarrow$ k is a prime generator for $6k + 1$ pattern

In other terms: for a given natural number $k > 0$:

$$k \notin \left\{ \begin{matrix} \text{natural numbers n, such that} \\ \exists \text{x, y natural numbers} \\ \text{suth that:} \\ n = \ 6xy + x - y \end{matrix} \right\} \Leftrightarrow 6k - 1 \text{ is a prime}$$

$$k \notin \left\{ \begin{matrix} \text{natural numbers n, such that} \\ \exists \text{x, y natural numbers} \\ \text{such that:} \\ n = \ 6xy + x + y \text{ or } \ n = \ 6xy - x - y \end{matrix} \right\} \Leftrightarrow 6k + 1 \text{ is a prime}$$

$$\text{where } k, x, y, n \geq 1 \text{ are all natural numbers}$$

In summary, seeking for 6k-1 like primes, comes to identifying prime generators coefficients for 6k-1 pattern, and seeking for 6k+1 like primes, comes to identifying 6k+1 prime generators coefficients. ([5]- KRAFFT, 1798)

From what is stated above, the 2 corollaries below are derived

1. **Corollary-1:**

$$\begin{gathered} \text{k is a prime generator for } 6k - 1 \text{ pattern} \\ \Leftrightarrow \\ \forall\, x > \ 0 \text{ natural number} \\ \text{k-x } \underline{\textbf{is not}} \text{ a multiple of } \ 6x - 1 \\ \text{where } k, x \geq 1 \text{ are natural numbers} \end{gathered}$$

<u>Proof of Corollary-1</u>**:**

- **First implication**:

$$\begin{aligned} &\text{k is a prime generator for } 6k - 1 \text{ pattern} \Rightarrow \ \text{k cannot be fitting in SET1 pattern} \\ &\Rightarrow \nexists\, x, y \ > 0 \ \text{ natural numbers, such that } \ k = \ 6xy + x - y \\ &\Rightarrow \ \nexists\, x, y \ > \ 0 \text{ natural numbers, such that } k - x = y(6x - 1) \\ &\Rightarrow \ \nexists \ x > \ 0 \text{ natural number, such that } k - x \text{ is a multiple of } 6x - 1 \\ &\Rightarrow \ \forall\, x \ > \ 0 \ \text{ natural number, } k - x \text{ is not a multiple of } 6x - 1 \end{aligned}$$

- **Second implication (**using contraposition**):**

$$\begin{aligned} &\text{k-x is a multiple of } \ 6x - 1 \Rightarrow \exists \text{y natural number, such that } \ k - x = y(6x - 1) \\ &\Rightarrow \ \exists \text{y natural number, such that } k = \ 6xy + x - y \\ &\Rightarrow \ \ \text{k is fitting in SET1 pattern} \\ &\Rightarrow \text{k is not a prime generator for } 6k - 1 \text{ pattern} \end{aligned}$$

$$\text{where } k, x, y \geq 1 \text{ are all natural numbers}$$

2. **Corollary-2:**

$$\begin{gathered} \text{k is a prime generator for } 6k + 1 \text{ pattern} \\ \Leftrightarrow \\ \forall\, x > 0 \text{ natural number: } [k - x \ \underline{\textbf{is not}} \text{ a multiple of } 6x + 1 \ \textbf{and} \ k + x \ \underline{\textbf{is not}} \text{ multiple of } 6x - 1] \end{gathered}$$

$$\text{where } k, x, y \geq 1 \text{ are all natural numbers}$$

Proof of Corollary-2:

- **First implication**:
  k is a prime generator for $6k + 1$ pattern $\Rightarrow$ k cannot be fitting in SET2 pattern

$$\Rightarrow \nexists\, x, y > 0 \text{ natural numbers, such that } k = 6xy + x + y \text{ or } k = 6xy - x - y$$
$$\Rightarrow \nexists\, x, y > 0 \text{ natural numbers, such that } k - x = y(6x + 1) \text{ or } k + x = y(6x - 1)$$
$$\Rightarrow \nexists\, x > 0 \text{ natural number: , such that}$$
$$[k - x \text{ is a multiple of } 6x + 1 \text{ or } k + x \text{ is a multiple of } 6x - 1]$$
$$\Rightarrow \forall\, x > 0 \text{ natural number: } [\, k - x \text{ \underline{\textbf{is not}} a multiple of } 6x + 1 \text{ and } k + x \text{ \underline{\textbf{is not}} multiple of } 6x - 1]$$

- **Second implication (**using contraposition**):**
  $k - x$ **is** a multiple of $6x + 1$ **or** $k + x$ **is** multiple of $6x - 1$

$$\Rightarrow \exists y \text{ natural number, such that } k - x = y(6x + 1)$$
$$\text{or } \exists z \text{ natural number, such that } k + x = z(6x - 1)$$
$$\Rightarrow \exists y \text{ natural number, such that } k = 6xy + x + y$$
$$\text{or } \exists z \text{ natural number, such that } k = 6xz - x - z$$
$$\Rightarrow \text{ k is fitting in SET2 patterns}$$
$$\Rightarrow \text{k is not a prime generator for } 6k + 1 \text{ pattern}$$

where $k, x, y, z \geq 1$ are all natural numbers

V. <u>**The structure of even numbers from the perspective of multiples of 3**</u>

1. Structure of natural Numbers:

On one hand, from the perspective of multiples of 3, any natural number $A > 4$ can be expressed as follows:

$$A = 3a - 1 \text{ or } A = 3a \text{ or } A = 3a + 1, \text{ where } a \geq 2 \text{ natural number.}$$

Indeed, the sequence-pattern: $[3a - 1, 3a, 3a + 1]$ repeats itself endlessly within the set of natural numbers:

$$\{\, [5,6,7] \;,\; [\ldots] \;,\; [101,102,103] \;,\; [\ldots] \,\} \quad \sim \quad \{\, [3a - 1, 3a, 3a + 1] \,,\; [\ldots] \,,\; [3a - 1, 3a, 3a + 1] \,,\; [\ldots] \,\}.$$

So, any natural number $A > 4$ must fall into one of these 3 categories: “3a-1" category, or"3a" category, or "3a + 1" category.

Consequently, by considering these three categories, all natural numbers > 4 are encompassed

2. Structure of Even natural Numbers:

On the other hand, by definition, an even natural number $N \geq 10$ in its general form can be expressed as: $N = 2A$, where $A > 4$ natural number

Considering the 3 categories introduced above, every even natural number ≥ 10, denoted as $N = 2A$, falls into one of three categories inherited from the natural number A it represents:

$$N = 2(3a - 1) \text{ or } N = 2(3a) \text{ or } N = 2(3a + 1), \text{ where } N \geq 10,\ a \geq 2 \text{ natural numbers}$$

The sequence-pattern: $[2(3a - 1), 2(3a), 2(3a + 1)]$ continues endlessly within the set of even natural numbers (≥ 10) and covers all of them:

$$\{\, [10,12,14] \;,\; [\ldots] \;,\; [\ldots] \,\} \quad \sim \quad \{\, [2(3a - 1), 2(3a), 2(3a + 1)] \;,\; [\ldots] \,,\; [\ldots] \,\}.$$

Finally, by considering these three categories, all even natural numbers ≥ 10 are encompassed

In this paper, modular arithmetic notation will be employed to examine the classification of a natural number A into each of these three categories; as follows:

- $A \equiv -1[3]$ for A belonging to "3a-1" category
- $A \equiv +1[3]$ for A belonging to "3a+1" category
- $A \equiv 0[3]$ for A belonging to "3a" category

VI. Introduction of the *ad-hoc Sieve notation* and the Preliminary propositions

1. Ad-hoc sieve notation:

In the propositions in this section, we'll use an *ad-hoc* sieve notation, introduced here, to filter numbers from a set based on a given condition expressed in the formulation. Within the context of the proof, the ad-hoc sieve notation will be utilized to indicate the existence and provide the form of the **B** parameters discussed in the proof.

The sieve notation comes in 2 types:

1. $\{\mathbf{0 \leq B < a, B \not\equiv x[n]\ for\ B \leq z}\}$**:** means in the initial range of values of B bound by **0** and **a** ,the numbers that are $\leq$ **z** and are congruent to **x** modulo **n** are sieved (excluded) from the set
   In other terms: $\{\mathbf{0 \leq B < a, B \not\equiv x[n]\ for\ B \leq z}\} = \{\mathbf{0 \leq B < a}\} \setminus \{\mathbf{B \equiv x[n]\ and\ B \leq z}\}$

   → **Example [02]:**
   $B \in \{0 \leq B < 10, B \not\equiv 1[5] \text{ for } B \leq 5\}$: means that **1 is sieved (**1 is congruent to 1 modulo 5 and 1$\leq$ 5**)** from the initial set of numbers below 10. The number **6 is not sieved (**regardless 6 is congruent to 1 modulo 5) since only numbers ≤ 5 are considered in the notation.
   Finally, values left in the set are: $B \in \{0,2,3,4,5,6,7,8,9\} = \{0 \leq B < 10, B \not\equiv 1[5] \text{ for } B \leq 5\}$

2. $\{\mathbf{0 \leq B < a, B \not\equiv x[n]\ for\ B \geq z}\}$: means that in the initial range of values of B bound by **0** and **a,** the numbers that are $\geq$ **z** and are congruent to **x** modulo **n** are sieved (excluded) from the set
   In other terms: $\{\mathbf{0 \leq B < a, B \not\equiv x[n]\ for\ B \geq z}\} = \{\mathbf{0 \leq B < a}\} \setminus \{\mathbf{B \equiv x[n]\ and\ B \geq z}\}$

   → **Example [03]:**
   $B \in \{0 \leq B < 10, B \not\equiv 1[5] \text{ for } B \geq 6\}$: means that **6 is sieved (**6 is congruent to 1 modulo 5 and $6 \geq 5$) from the initial set of numbers **strictly below 10**. The number **1 is not sieved** since only numbers ≥ 6 are considered in the notation.
   Finally, values left are $B \in \{0,1,2,3,4,5,7,8,9\} = \{0 \leq B < 10, B \not\equiv 1[5] \text{ for } B \geq 6\}$

<u>NB</u>: This paper presumes the non-emptiness of the set resulting from a sieve or combination of sieves. However, this assumption can be substantiated through the application of techniques used in sieve theory ([11]- Greaves, 2001).

2. Proposition 1.1:

For every natural number $a \geq 2$,
For every natural number $B < a$ of the same parity as a:

If $B \in \{0 \leq B < a, B \not\equiv (a - 2n)[6n - 1] \text{ for } B \leq a - 2(7n - 1)\}$

Then $\left(\frac{a - B}{2} - n\right)$ is not a multiple of $(6n - 1)$; for all natural numbers $n \geq 1$

N.B.: The non emptiness of the set $\{0 \leq B < a, B \not\equiv (a - 2n)[6n - 1] \text{for } B \leq a - 2(7n - 1)\}$ is presumed

→ **Example [04]**: for $a = 17$:
Regarding the assertion, $\left(\frac{17-B}{2} - n\right)$ *is not a multiple of* $(6n - 1)$, *for all natural numbers* $n \geq 1$
n has only one value to be checked: $n = 1$. For the rest of n values > 1, the assertion is trivial

Pick B as any odd number (same parity as 17) $\in \{0 \leq B < 17, B \not\equiv (17 - 2)\ [5\ ] \text{ for } B \leq 17 - 2(7 - 1)\}$

$$\Leftrightarrow$$

$$\text{Any odd number } B \in \{0 \leq B < 17, B \not\equiv 0\ [5\ ] \text{ for } B \leq 5\}$$

$$\Leftrightarrow$$

$$\text{Pick B as any number } B \in \{1,3,\cancel{5},7,9,11,13,15\}$$

(here 5 is sieved because: $5 \equiv 0\ [5\ ]$ and $5 \leq 5$, emphasized with 5 with a strike)

So, left values of B are: any odd number less than 17 and different than 5 (the only sieved value)
The statement of the proposition is confirmed using the values: a=17, n=1 and the left B values:

- for $B = 1, \left(\frac{17-1}{2} - 1 = 7\right)$ is not a multiple of $(6 - 1 = 5)$
- for $B = 3, \left(\frac{17-3}{2} - 1 = 6\right)$ is not a multiple of $(6 - 1 = 5)$
- for $B = 7, \left(\frac{17-7}{2} - 1 = 4\right)$ is not a multiple of $(6 - 1 = 5)$
- for $B = 9, \left(\frac{17-9}{2} - 1 = 3\right)$ is not a multiple of $(6 - 1 = 5)$
- for $B = 11, \left(\frac{17-11}{2} - 1 = 2\right)$ is not a multiple of $(6 - 1 = 5)$
- for $B = 13, \left(\frac{17-13}{2} - 1 = 1\right)$ is not a multiple of $(6 - 1 = 5)$
- for $B = 15, \left(\frac{17-15}{2} - 1 = 0\right)$ is not a multiple of $(6 - 1 = 5)$

Proof of proposition 1.1:

Let be a, B and n natural numbers such that $a \geq 2, B < a$ and $n \geq 1$

On the first hand, knowing that $n \geq 1, \left(\frac{a-B}{2}\right)$ is a natural number only if B and is of the same parity as a

→ Thus, Constraint 1: B is of the same parity as a

On the second hand, by definition: $\left(\frac{a-B}{2} - n\right)$ is not a multiple of $(6n - 1) \Leftrightarrow \left(\frac{a-B}{2} - n\right) \not\equiv 0\ [6n - 1]$

$\Leftrightarrow B \not\equiv (a - 2n)\ [6n - 1\ ]$

So, the values of B satisfying $\left(\frac{a-B}{2} - n\right)$ *is not a multiple of* $(6n - 1)$ are only B values that are non-congruent with $(a - 2n)$ modulo $[6n - 1\ ]$: $B \not\equiv (a - 2n)\ [6n - 1\ ]$

→ Thus, Constraint 2 : $B \not\equiv (a - 2n)\ [6n - 1\ ]$

On the third hand, to be more selective on Constraint 2, two cases need to be discussed:

- **Case 1:** $\left(\frac{\mathbf{a}}{\mathbf{2}} - \mathbf{n}\right) < \mathbf{6n - 1}$:

  in this case $\left(\frac{a-B}{2} - n\right)$ can never be a multiple of $6n - 1$ (B is positive which keeps $\left(\frac{a-B}{2} - n\right)$ less than $6n - 1$)
  Thus, in this case all natural numbers B such that $0 \leq B < a$ satisfy the expression $\left(\frac{a-B}{2} - n\right)$ *is not a multiple of* $(6n - 1)$
- **Case 2:** $\left(\frac{\mathbf{a}}{\mathbf{2}} - \mathbf{n}\right) \geq \mathbf{6n - 1}$:

  in this case $\left(\frac{a-B}{2} - n\right) \geq 6n - 1 \Leftrightarrow B \leq a - 2(7n - 1)$

→ Scope of Constraint 2: The discussed 2 cases above resulting on targeting only B values: $B \leq a - 2(7n - 1)$ with the Constraint 2

Combining the constraints above to finely choose B such that:

- [Constraint 1]: $B < a$ and the same parity as a and
- [Constraint 2]: $B \not\equiv (a - 2n)\ [6n - 1\ ]$
- [Scope of Constraint 2]: For only constrained B values: $B \leq a - 2(7n - 1)$

To sum it up using the sieve notation introduced previously to encapsulate the constraints 2:

- For evey $B < a$ of the same parity as a:
- If $\mathbf{B} \in \{\mathbf{0} \leq \mathbf{B} < \mathbf{a}, \mathbf{B} \not\equiv (\mathbf{a} - \mathbf{2n})[\mathbf{6n} - \mathbf{1}]\mathbf{for}\ \mathbf{B} \leq \mathbf{a} - \mathbf{2}(\mathbf{7n} - \mathbf{1})\}$

  Then $\left(\frac{a - B}{2} - n\right)$ is not a multiple of $(6n - 1)$; for all natural numbers $n \geq 1$

Also, $\left(\frac{a-B}{2} - n\right) \geq 6n - 1$ means that the upper limit of n is $\frac{a+2}{14}$. Beyond this limit, the assertion is trivial

$n \leq \frac{a-B+2}{14} \leq \frac{a-0+2}{14} = \frac{a+2}{14}$ (because maximum n requires minimum B which is 0)

NB: The non-emptiness of the set $\{0 \leq B < a, \; B \not\equiv (a-2n)\,[6n-1]$ for $B \leq a-2(7n-1)\}$ is presumed .

3. Proposition 1.2:

For every natural number $a \geq 2$,
For every natural number $B < a$ of the same parity as a:

If $B \in \{0 \leq B < a, B \not\equiv (2n-a)[6n-1]$ for $B \geq 2(7n-1)-a\}$
Then $\left(\frac{a+B}{2} - n\right)$ is not a multiple of $(6n-1)$; for all natural numbers $n \geq 1$

N.B.: The non emptiness of the set $\{0 \leq B < a, B \not\equiv (2n-a)[6n-1]$ for $B \geq 2(7n-1)-a\}$ is presumed

→ **Example [05]**: for $a = 20$:

Regarding the assertion, $\left(\frac{20+B}{2} - n\right)$ *is not a multiple of* $(6n-1)$, *for all natural numbers* $n \geq 1$
n has only 2 values to be checked: $n = 1$ and $n = 2$. For the rest of n values > 2, the assertion is trivial

Pick B as any even number (same parity as 20) $\in \{0 \leq B < 20, B \not\equiv (2-20)\,[5]$ for $B \geq 2(7-1)-20\} \cap$
$\{0 \leq B < 20, B \not\equiv (4-20)\,[11]$ for $B \geq 2(14-1)-20\}$

$$\Leftrightarrow$$

Any even number $B \in \{0 \leq B < 20, B \not\equiv 2\,[5]$ for $B \geq -8\} \cap \{0 \leq B < 20, B \not\equiv 6\,[11]$ for $B \geq 6\}$

$$\Leftrightarrow$$

$$B \in \{0,\cancel{2},4,\cancel{6},8,10,\cancel{12},14,16,18\}$$

(here 2,6, and 12 are sieved, emphasized with a strike on numbers)

So left values of B: any even number less than 20 and different than 2, 6, and 12
The statement of the proposition is confirmed using the values: a=20, n = 1, 2 and left B values:

- for $B = 0$, $\left(\frac{20+0}{2} - 1 = 9\right)$ is not a multiple of 5 and $\left(\frac{20+0}{2} - 2 = 8\right)$ is not a multiple of 11
- for $B = 4$, $\left(\frac{20+4}{2} - 1 = 11\right)$ is not a multiple of 5 and $\left(\frac{20+4}{2} - 2 = 10\right)$ is not a multiple of 11
- for $B = 8$, $\left(\frac{20+8}{2} - 1 = 13\right)$ is not a multiple of 5 and $\left(\frac{20+8}{2} - 2 = 12\right)$ is not a multiple of 11
- for $B = 10$, $\left(\frac{20+10}{2} - 1 = 14\right)$ is not a multiple of 5 and $\left(\frac{20+10}{2} - 2 = 13\right)$ is not a multiple of 11
- for $B = 14$, $\left(\frac{20+14}{2} - 1 = 16\right)$ is not a multiple of 5 and $\left(\frac{20+14}{2} - 2 = 15\right)$ is not a multiple of 11
- for $B = 16$ $\left(\frac{20+1}{2} - 1 = 17\right)$ is not a multiple of 5 and $\left(\frac{20+16}{2} - 2 = 16\right)$ is not a multiple of 11
- for $B = 18$, $\left(\frac{20+18}{2} - 1 = 18\right)$ is not a multiple of 5 and $\left(\frac{20+18}{2} - 2 = 17\right)$ is not a multiple of 11

<u>Proof of proposition 1.2:</u>

Let be a, B and n natural numbers such that $a \geq 2$, B<a and $n \geq 1$
On the first hand, $\left(\frac{a+B}{2}\right)$ is a natural number only if B is of the same parity as a

→ Thus, <u>Constraint 1</u>: B is of the same parity as a

On the second hand, by definition: $\left(\frac{a+B}{2} - n\right)$ is not a multiple of $(6n-1) \Leftrightarrow \left(\frac{a+B}{2} - n\right) \not\equiv 0\,[6n-1]$
$\Leftrightarrow B \not\equiv (\mathbf{2n} - \mathbf{a})\,[6n-1]$

So, the values of B satisfying $\left(\frac{a+B}{2} - n\right)$ *is not a multiple of* $(6n-1)$ are only B values that are non-congruent with $(\mathbf{2n} - \mathbf{a})$ modulo $[6n-1]$: $B \not\equiv (\mathbf{2n} - \mathbf{a})\,[6n-1]$

→ Thus, <u>Constraint 2</u>: $B \not\equiv (\mathbf{2n} - \mathbf{a})\,[6n-1]$

On the third hand, to be more selective on Constraint 2, two cases need to be discussed:

- **Case 1:** $\left(\frac{\mathbf{a+B}}{\mathbf{2}} - \mathbf{n}\right) < \mathbf{6n-1}$:

  in this case $\left(\frac{a+B}{2} - n\right)$ can never be a multiple of $6n-1$

  Thus, in this case all natural numbers B such that $0 \leq B < a$ satisfy the expression $\left(\frac{a+B}{2} - n\right)$ *is not a multiple of* $(6n-1)$

- **Case 2:** $\left(\frac{\mathbf{a+B}}{\mathbf{2}} - \mathbf{n}\right) \geq \mathbf{6n-1}$:

  in this case $\left(\frac{a+B}{2} - n\right) \geq 6n-1 \Leftrightarrow B \geq 2(7n-1) - a$

→ Scope of Constraint 2: The discussed 2 cases above resulting on targeting only B values: $B \geq 2(7n-1) - a$ with the Constraint 2

Combining the constraints above to finely choose B such that:

- [Constraint 1]: B is of the same parity as a and
- [Constraint 2]: $B \not\equiv (\mathbf{2n-a})\ [6n-1\,]$
- [Scope of Constraint 2]: For only constrained B values: $B \geq 2(7n-1) - a$

To sum it up using the sieve notation introduced previously to encapsulate the constraints 2:

- For evey B of the same parity as a:
- If $\mathbf{B} \in \{\mathbf{0} \leq \mathbf{B} < \mathbf{a}, \mathbf{B} \not\equiv (\mathbf{2n-a})[\mathbf{6n-1}\,]\ \mathbf{for\ B} \geq \mathbf{2(7n-1) - a}\,\}$

  then $\left(\frac{a+B}{2} - n\right)$ is not a multiple of $(6n-1)$; for all natural numbers $n \geq 1$

And also, $\left(\frac{a+B}{2} - n\right) \geq 6n-1$ means that the upper limit of n is: $\frac{2a+1}{14}$. Beyond this value, the assertion is trivial.

$$n \leq \frac{a+B+2}{14} \leq \frac{a+a-1+2}{14} = \frac{2a+1}{14} \quad \text{(maximum n requires maximum B which is a-1)}$$

NB: The non-emptiness of the set $\{0 \leq B < a, B \not\equiv (2n-a)[6n-1\,]\ \text{for}\ B \geq 2(7n-1) - a\,\}$ is presumed

4. Proposition 1.3:

For every natural number $a \geq 2$,
For every natural number $B < a$ of the same parity as a:

If $B \in \{0 \leq B < a, B \not\equiv (a-2n)[6n+1]\ \ \text{for}\ B \leq a - 2(7n+1)\}$

then $\left(\frac{a-B}{2} - n\right)$ is not a multiple of $(6n+1)$ for all natural numbers $n \geq 1$

N.B.: The non emptiness of the set $\{0 \leq B < a, B \not\equiv (a-2n)[6n+1]\ \ \text{for}\ B \leq a - 2(7n+1)\}$ is presumed

→ **Example [06]**: for $a = 38$

Regarding the assertion, $\left(\frac{38-B}{2} - n\right)$ *is not a multiple of* $(6n+1)$, *for all natural numbers* $n \geq 1$

n has only 2 values to be checked: $n = 1{,}2$. For the rest of n values > 2, the assertion is trivial

Pick B as any even number (same parity as 38) $\in \{0 \leq B < 38, B \not\equiv (38-2)\ [7\,]\ \text{for}\ B \leq 38 - 2(7+1)\,\} \cap \{0 \leq B < 38, B \not\equiv (38-4)[13]\ \text{for}\ B \leq 38 - 2(14+1)\}$

$$\Leftrightarrow$$

Any even number $B \in \{0 \leq B < 38, B \not\equiv 1\ [7\,]\ \text{for}\ B \leq 22\} \cap \{0 \leq B < 38, B \not\equiv 8[13]\ \text{for}\ B \leq 8\}$

$$\Leftrightarrow$$

$$B \in \{0,2,4,6,\cancel{8},10,12,14,16,18,20,\cancel{22},24,26,28,30,32,34,36\}$$

(here 8 and 12 are sieved, emphasized with a strike on numbers)

So, left values of B are: any even number less than 38 and different than 8 and 22 is fitting
The statement of the proposition is confirmed using the values: a=38, n = {1, 2}, and the left B values.

- for $B = 0$, $\left(\frac{38-0}{2} - 1 = 18\right)$ is not a multiple of 7 and $\left(\frac{38-0}{2} - 2 = 17\right)$ is not a multiple of 13

- for $B = 2, \left(\frac{38-2}{2} - 1 = 17\right)$ is not a multiple of 7 and $\left(\frac{38-2}{2} - 2 = 16\right)$ is not a multiple of 13
- for $B = 4, \left(\frac{38-4}{2} - 1 = 16\right)$ is not a multiple of 7 and $\left(\frac{38-4}{2} - 2 = 15\right)$ is not a multiple of 13
- for $B = 6, \left(\frac{38-6}{2} - 1 = 15\right)$ is not a multiple of 7 and $\left(\frac{38-6}{2} - 2 = 14\right)$ is not a multiple of 13
- for $B = 10, \left(\frac{38-10}{2} - 1 = 13\right)$ is not a multiple of 7 and $\left(\frac{38-10}{2} - 2 = 12\right)$ is not a multiple of 13
- for $B = 12, \left(\frac{38-12}{2} - 1 = 12\right)$ is not a multiple of 7 and $\left(\frac{38-1}{2} - 2 = 11\right)$ is not a multiple of 13
- for $B = 14, \left(\frac{38-14}{2} - 1 = 11\right)$ is not a multiple of 7 and $\left(\frac{38-14}{2} - 2 = 10\right)$ is not a multiple of 13
- for $B = 16, \left(\frac{38-16}{2} - 1 = 10\right)$ is not a multiple of 7 and $\left(\frac{38-16}{2} - 2 = 9\right)$ is not a multiple of 13
- for $B = 18, \left(\frac{38-18}{2} - 1 = 9\right)$ is not a multiple of 7 and $\left(\frac{38-18}{2} - 2 = 8\right)$ is not a multiple of 13
- for $B = 20, \left(\frac{38-20}{2} - 1 = 8\right)$ is not a multiple of 7 and $\left(\frac{38-20}{2} - 2 = 7\right)$ is not a multiple of 13
- for $B = 24, \left(\frac{38-2}{2} - 1 = 6\right)$ is not a multiple of 7 and $\left(\frac{38-24}{2} - 2 = 5\right)$ is not a multiple of 13
- for $B = 26, \left(\frac{38-26}{2} - 1 = 5\right)$ is not a multiple of 7 and $\left(\frac{38-26}{2} - 2 = 4\right)$ is not a multiple of 13
- for $B = 28, \left(\frac{38-28}{2} - 1 = 4\right)$ is not a multiple of 7 and $\left(\frac{38-28}{2} - 2 = 3\right)$ is not a multiple of 13
- for $B = 30, \left(\frac{38-30}{2} - 1 = 3\right)$ is not a multiple of 7 and $\left(\frac{38-30}{2} - 2 = 2\right)$ is not a multiple of 13
- for $B = 32, \left(\frac{38-32}{2} - 1 = 2\right)$ is not a multiple of 7 and $\left(\frac{38-32}{2} - 2 = 1\right)$ is not a multiple of 13
- for $B = 34, \left(\frac{38-34}{2} - 1 = 1\right)$ is not a multiple of 7 and $\left(\frac{38-34}{2} - 2 = 0\right)$ is not a multiple of 13
- for $B = 36, \left(\frac{38-36}{2} - 1 = 0\right)$ is not a multiple of 7 and $\left(\frac{38-36}{2} - 2 = -1\right)$ is not a multiple of 13

Proof of proposition 1.3:

Let be a, B and n natural numbers such that $a \geq 2$, B<a and $n \geq 1$

On the first hand, knowing that $n \geq 1$, $\left(\frac{a-B}{2}\right)$ is a natural number only if B is of the same parity as a

→ Thus, Constraint 1: B of the same parity as a

On the second hand, by definition: $\left(\frac{a-B}{2} - n\right)$ is not a multiple of $(6n+1)$ $\Leftrightarrow$ $\left(\frac{a-B}{2} - n\right) \not\equiv 0\ [6n+1]$

$\Leftrightarrow$ $\mathbf{B \not\equiv (a - 2n)\ [6n + 1]}$

So, the values of B satisfying $\left(\frac{a-B}{2} - n\right)$ *is not a multiple of* $(6n+1)$ are only B values that are non-congruent with $(a - 2n)$ modulo $[6n + 1]$: $\mathbf{B \not\equiv (a - 2n)\ [6n + 1]}$

→ Thus, Constraint 2: $B \not\equiv (a - 2n)\ [6n + 1]$

On the third hand, to be more selective on Constraint 2, two cases need to be discussed:

- **Case 1:** $\left(\frac{a}{2} - n\right) < 6n + 1$**:**

  in this case $\left(\frac{a-B}{2} - n\right)$ can never be a multiple of $6n - 1$ (B is positive which keeps $\left(\frac{a-B}{2} - n\right)$ lesser than $6n + 1$)

  Thus, in this case all natural numbers B such that $0 \leq B < a$ satisfy the expression $\left(\frac{a-B}{2} - n\right)$ *is not a multiple of* $(6n+1)$

- **Case 2:** $\left(\frac{a}{2} - n\right) \geq 6n + 1$:

  in this case $\left(\frac{a-B}{2} - n\right) \geq 6n + 1 \Leftrightarrow B \leq a - 2(7n + 1)$

→ Scope of Constraint 2: The discussed 2 cases above resulting on targeting only B values: $B \leq a - 2(7n + 1)$ with the Constraint 2

Combining the constraints above to finely choose B such that:

- [Constraint 1]: B is of the same parity as a and
- [Constraint 2]: $B \not\equiv (a - 2n)\ [6n + 1]$

- [Scope of Constraint 2]: For only constrained B values: $B \leq a - 2(7n+1)$

To sum it up using the sieve notation introduced previously to encapsulate the constraints 2:

- For evey B is of the same parity as a:
- If $\mathbf{B} \in \{\mathbf{0} \leq \mathbf{B} < \mathbf{a}, \mathbf{B} \not\equiv (\mathbf{a} - \mathbf{2n})[\mathbf{6n} + \mathbf{1}] \ \mathbf{for\ B} \leq \mathbf{a} - \mathbf{2}(\mathbf{7n} + \mathbf{1})\}$

  then $\left(\frac{a - B}{2} - n\right)$ is not a multiple of $(6n+1)$; for all natural numbers $n \geq 1$

And also, $\left(\frac{a-B}{2} - n\right) \geq 6n+1$ means that the upper limit of n is $\frac{a-2}{14}$. Beyond this value, the assertion is trivial.

$$n \leq \frac{a-B-2}{14} \leq \frac{a-0-2}{14} = \frac{a-2}{14} \quad \text{(because maximum n requires minimum B which is 0)}$$

NB: The non-emptiness of the set $\{0 \leq B < a, B \not\equiv (a-2n)[6n+1] \ \text{ for } B \leq a - 2(7n+1)\}$ is presumed.

5. Proposition 1.4:

For every natural number $a \geq 2$,
For every natural number $B < a$ of the same parity as a:

If $B \in \{0 \leq B < a, B \not\equiv (2n-a)[6n+1] \text{ for } B \geq 2(7n+1) - a\}$
Then $\left(\frac{a+B}{2} - n\right)$ **is not** a multiple of $(6n+1)$, for all natural numbers $n \geq 1$

N.B.: The non emptiness of the set $\{0 \leq B < a, B \not\equiv (2n-a)[6n+1] \text{ for } B \geq 2(7n+1) - a\}$ is presumed

→ **Example [07]**: for $a = 38$

Regarding the assertion, $\left(\frac{38+B}{2} - n\right)$ *is not a multiple of* $(6n+1)$, *for all natural numbers* $n \geq 1$

n has only five values to be checked: $n = 1,2,3,4,5$. For the rest of n values > 5, the assertion is trivial

Pick B as any even number (same parity as 38) $\in \{0 \leq B < 38, B \not\equiv (2-38)\ [7] \text{ for } B \geq 2*8-38\} \cap$
$\{0 \leq B < 38, B \not\equiv (4-38)[13] \text{ for } B \geq 2*15-38\} \cap$
$\{0 \leq B < 38, B \not\equiv (6-38)[19] \text{ for } B \geq 2*22-38\} \cap$
$\{0 \leq B < 38, B \not\equiv (8-38)\ [25] \text{ for } B \geq 2*29-38\} \cap$
$\{0 \leq B < 38, B \not\equiv (10-38)[31] \text{ for } B \geq 2*36-38\}$

$\Leftrightarrow$

B is even and
$B \in \{0 \leq B < 38, B \not\equiv 6\ [7] \text{ for } B \geq -22\} \cap$
$\{0 \leq B < 38, B \not\equiv 5[13] \text{ for } B \geq -8\} \cap$
$\{0 \leq B < 38, B \not\equiv 6[19] \text{ for } B \geq 6\} \cap$
$\{0 \leq B < 38, B \not\equiv 20[25] \text{ for } B \geq 20\} \cap$
$\{0 \leq B < 38, B \not\equiv 3[31] \text{ for } B \geq 34\}$

$B \in \{0,2,4,\cancel{6},8,10,12,14,16,\cancel{18},\cancel{20},22,24,26,28,30,32,\cancel{34},36\}$
(here 6, 18, 20, 34 are sieved, emphasized with a strike on numbers)

So left values of B: any even number < 38 and different than 6, 18, 20, 34 is fitting.

The statement of the proposition is confirmed using the values: a=38, n=1,2,3,4,5 and left B values.
Below is the numerical check for B=4

- $\left(\frac{38+4}{2} - 1 = 20\right)$ is not a multiple of $(6+1=7)$, $\quad n = 1$
- $\left(\frac{38+4}{2} - 2 = 19\right)$ is not a multiple of $(12+1=13)$, $\quad n = 2$
- $\left(\frac{38+4}{2} - 3 = 18\right)$ is not a multiple of $(18+1=19)$, $\quad n = 3$

- $\left(\frac{38+4}{2} - 4 = 17\right)$ is not a multiple of $(24 + 1 = 25)$, $n = 4$
- $\left(\frac{38+4}{2} - 5 = 16\right)$ is not a multiple of $(30 + 1 = 31)$, $n = 5$

The same can be verified all the B left values {0,2,4 (checked),8,10,12,14,16,22,24,26,28,30,32,36}

Proof of proposition 1.4:

Let be a, B and n natural numbers such that $a \geq 2$, B<a and $n \geq 1$

On the first hand, $\left(\frac{a+B}{2}\right)$ is a natural number only if B is of the same parity as a

→ Thus, Constraint 1: B is of the same parity as a

On the second hand, by definition $\left(\frac{a+B}{2} - n\right)$ is not a multiple of $(6n + 1)$ $\Leftrightarrow$ $\left(\frac{a+B}{2} - n\right) \not\equiv 0\ [6n + 1]$

$\Leftrightarrow$ $\mathbf{B \not\equiv (2n - a)[6n + 1\,]}$

So, the values of B satisfying $\left(\frac{a+B}{2} - n\right)$ *is not a multiple of* $(6n + 1)$ are only B values that are non-congruent with $(\mathbf{2n - a})$ modulo $[6n + 1\,]$: $\mathbf{B \not\equiv (2n - a)[6n + 1]}$

→ Thus, Constraint 2: $\mathbf{B \not\equiv (2n - a)[6n + 1]}$

On the third hand, to be more selective on Constraint 2, two cases need to be discussed:

- **Case 1:** $\left(\frac{\mathbf{a+B}}{\mathbf{2}} - \mathbf{n}\right) < \mathbf{6n + 1}$**:**

  in this case $\left(\frac{a+B}{2} - n\right)$ can never be a multiple of $6n + 1$

  Thus, in this case all natural numbers B such that $0 \leq B < a$ satisfy the expression $\left(\frac{a+B}{2} - n\right)$ *is not a multiple of* $(6n + 1)$

- **Case 2:** $\left(\frac{\mathbf{a+B}}{\mathbf{2}} - \mathbf{n}\right) \geq \mathbf{6n + 1}$:

  in this case $\left(\frac{a+B}{2} - n\right) \geq 6n + 1 \Leftrightarrow B \geq 2(7n + 1) - a$

→ Scope of Constraint 2: The discussed 2 cases above resulting on targeting only B values: $B \geq 2(7n + 1) - a$ with the Constraint 2

Combining the constraints above to finely choose B such that:

- [Constraint 1]: B is of the same parity as a and
- [Constraint 2]: $B \not\equiv (\mathbf{2n - a})\ [6n + 1\,]$
- [Scope of Constraint 2]: For only constrained B values $B \geq 2(7n - 1) - a$

To sum it up using the sieve notation introduced previously to encapsulate the constraints 2:

- For evey B is of the same parity as a:
- If $\mathbf{B} \in \{\mathbf{0 \leq B < a, B \not\equiv (2n - a)[6n + 1\,]\ for\ B \geq 2(7n - 1) - a}\,\}$

  then $\left(\frac{a + B}{2} - n\right)$ is not a multiple of $(6n + 1)$; for all natural numbers $n \geq 1$

And also, $\left(\frac{a+B}{2} - n\right) \geq 6n + 1$ means that the upper limit of n is: $\frac{2a-3}{14}$. Beyond this value, the assertion is trivial.

$$n \leq \frac{a+B-2}{14} \leq \frac{a+a-1-2}{14} = \frac{2a-3}{14} \quad \text{(maximum n requires maximum B which is a-1)}$$

NB: The non-emptiness of the set $\{0 \leq B < a, B \not\equiv (2n - a)[6n + 1\,]$ for $B \geq 2(7n + 1) - a\}$ is presumed.

6. Proposition 1.5:

For every natural number $a \geq 2$,

For every natural number $B < a$ of the same parity as a:

If $B \in \{0 \leq B < a, B \not\equiv (a+2n)[6n-1] \text{ for } B \leq a-2(5n-1)\}$
Then $\left(\frac{a-B}{2} + n\right)$ is not a multiple of $(6n-1)$ for all numbers $n \geq 1$

N.B.: The non-emptiness of the set $\{0 \leq B < a, B \not\equiv (a+2n)[6n-1] \text{ for } B \leq a-2(5n-1)\}$ is presumed

→ **<u>Example [08]</u>**: for $a = 21$

Regarding the assertion $\left(\frac{a-B}{2} + n\right)$ *is not a multiple of* $(6n-1)$, for all numbers $n \geq 1$
n has two possible values to be checked: 1, 2. For the rest of n values, the assertion is trivial.

Pick B as any odd number (same parity as 21) $\in \{0 \leq B < 21, B \not\equiv (21+2)\,[5] \text{ for } B \leq 21-2(5-1)\} \cap \{0 \leq B < 21, \{B \not\equiv (21+4)[11] \text{ for } B \leq 21-2(5*2-1)\}$

$$\Leftrightarrow$$

any odd number $B \in \{0 \leq B < 21, B \not\equiv 3\,[5] \text{ for } B \leq 13\} \cap \{0 \leq B < 21, B \not\equiv 3[11] \text{ for } B \leq 3\}$

$$\Leftrightarrow$$

$$B \in \{1,\cancel{3},5,7,9,11,\cancel{13},15,17,19\}$$

(here 3 and 13 are sieved, emphasized with a strike on numbers)

So, left values of B are: any odd number less than 21 and different than 3 and 13
The statement of the proposition is confirmed using the values: a=21, n=1,2 and the left B values:

- for $B = 1, \left(\frac{21-1}{2} + 1 = 11\right)$ is not a multiple of 5 and $\left(\frac{21-1}{2} + 2 = 12\right)$ is not a multiple of 11
- for $B = 5, \left(\frac{21-5}{2} + 1 = 9\right)$ is not a multiple of 5 and $\left(\frac{21-5}{2} + 2 = 10\right)$ is not a multiple of 11
- for $B = 7, \left(\frac{21-7}{2} + 1 = 8\right)$ is not a multiple of 5 and $\left(\frac{21-7}{2} + 2 = 9\right)$ is not a multiple of 11
- for $B = 9, \left(\frac{21-9}{2} + 1 = 7\right)$ is not a multiple of 5 and $\left(\frac{21-9}{2} + 2 = 8\right)$ is not a multiple of 11
- for $B = 11, \left(\frac{21-11}{2} + 1 = 6\right)$ is not a multiple of 5 and $\left(\frac{21-11}{2} + 2 = 7\right)$ is not a multiple of 11
- for $B = 15, \left(\frac{21-15}{2} + 1 = 4\right)$ is not a multiple of 5 and $\left(\frac{21-15}{2} + 2 = 5\right)$ is not a multiple of 11
- for $B = 17, \left(\frac{21-17}{2} + 1 = 3\right)$ is not a multiple of 5 and $\left(\frac{21-17}{2} + 2 = 4\right)$ is not a multiple of 11
- for $B = 19, \left(\frac{21-19}{2} + 1 = 2\right)$ is not a multiple of 5 and $\left(\frac{21-19}{2} + 2 = 3\right)$ is not a multiple of 11

<u>Proof of proposition 1.5:</u>

Let be a, B and n natural numbers such that $a \geq 2$, $B<a$ and $n \geq 1$
On the first hand, knowing that $n \geq 1$, $\left(\frac{a-B}{2}\right)$ is a natural number only if B is of the same parity as a

→ Thus, <u>Constraint 1</u>: B is of the same parity as a

On the second hand, by definition: $\left(\frac{a-B}{2} + n\right)$ is not a multiple of $(6n-1) \Leftrightarrow \left(\frac{a-B}{2} + n\right) \not\equiv 0\,[6n-1]$
$\Leftrightarrow \mathbf{B \not\equiv (a+2n)\,[6n-1]}$

So, the values of B satisfying $\left(\frac{a-B}{2} + n\right)$ *is not a multiple of* $(6n-1)$ are only B values that are non-congruent with $(a+2n)$ modulo $[6n-1]$: $B \not\equiv (a+2n)\,[6n-1]$

→ Thus, <u>Constraint 2</u> $B \not\equiv (a+2n)\,[6n-1]$

On the third hand, to be more selective on Constraint 2, two cases need to be discussed:

- **Case 1:** $\left(\frac{a}{2} + n\right) < 6n - 1$**:**
  in this case $\left(\frac{a-B}{2} + n\right)$ can never be a multiple of $6n-1$ (B is positive which keeps $\left(\frac{a-B}{2} + n\right)$ lesser than $6n-1$)
  Thus, in this case all natural numbers B such that $0 \leq B < a$ satisfy the expression
  $\left(\frac{a-B}{2} + n\right)$ *is not a multiple of* $(6n-1)$
- **Case 2:** $\left(\frac{a}{2} + n\right) \geq 6n - 1$:

in this case $\left(\frac{a-B}{2}+n\right) \geq 6n-1 \Leftrightarrow B \leq a-2(5n-1)$

→ Scope of Constraint 2: The discussed 2 cases above resulting on targeting only B values: $B \leq a-2(5n-1)$with the Constraint 2

Combining the constraints above to finely choose B such that:

- [Constraint 1]: B is of the same parity as a and
- [Constraint 2]: $B \not\equiv (a+2n)\ [6n-1]$
- [Scope of Constraint 2]: For only constrained B values: $B \leq a-2(5n-1)$

To sum it up using the sieve notation introduced previously to encapsulate the constraints 2:

- For evey B is of the same parity as a:
- If $\mathbf{B} \in \{\mathbf{0} \leq \mathbf{B} < \mathbf{a}, \mathbf{B} \not\equiv (\mathbf{a}+\mathbf{2n})\ [\mathbf{6n}-\mathbf{1}]$ **for** $\mathbf{B} \leq \mathbf{a}-\mathbf{2}(\mathbf{5n}-\mathbf{1})\}$

  Then $\left(\dfrac{a-B}{2}+n\right)$ is not a multiple of $(6n-1)$; for all natural numbers $n \geq 1$

And also, $\left(\frac{a-B}{2}+n\right) \geq 6n-1$ means that the upper limit of n is $\frac{a+2}{10}$. Beyond this value, the assertion is trivial

$$n \leq \frac{a-B+2}{10} \leq \frac{a-0+2}{10} = \frac{a+2}{10} \quad \text{(because maximum n requires minimum B which is 0)}$$

NB: The non-emptiness of the set $\{0 \leq B < a, B \not\equiv (a+2n)[6n-1]$ for $B \leq a-2(5n-1)\}$ is presumed.

7. Proposition 1.6:

For every natural number $a \geq 2$,
For every natural number $B < a$ of the same parity as a:

If $B \in \{0 \leq B < a, B \not\equiv -(a+2n)[6n-1]$ for $B \geq 2(5n-1)-a\}$
Then $\left(\frac{a+B}{2}+n\right)$ **is not** a multiple of $(6n-1)$, for all numbers $n \geq 1$

N.B.: The non emptiness of the set $\{0 \leq B < a, B \not\equiv -(a+2n)[6n-1]$ for $B \geq 2(5n-1)-a\}$ is presumed

→ **Example [09]**: for $a = 21$

Regarding the assertion $\left(\frac{21+B}{2}+n\right)$ ***is not*** *a multiple of* $(6n-1)$, *for all numbers* $n \geq 1$
n has four possible values to be checked: 1, 2, 3, 4 ($1 \leq n$): For the rest of n values, the assertion is trivial.

Pick B as any odd number (same parity as 21) and $\in \{0 \leq B < 21, B \not\equiv -(21+2)\ [5]$ for $B \geq 2*4-21\} \cap$

$$\{0 \leq B < 21, B \not\equiv -(21+4)[11] \text{ for } B \geq 2*9-21\} \cap$$
$$\{0 \leq B < 21, B \not\equiv -(21+6)[17] \text{ for } B \geq 2*14-21\} \cap$$
$$\{0 \leq B < 21, B \not\equiv -(21+8)[23] \text{ for } B \geq 2*19-21\}$$
$$\Leftrightarrow$$
B is odd and
$$B \in \{0 \leq B < 21, B \not\equiv 2\ [5] \text{ for } B \geq -13\} \cap$$
$$\{0 \leq B < 21, B \not\equiv 8\ [11] \text{ for } B \geq -3\} \cap$$
$$\{0 \leq B < 21, B \not\equiv 7\ [17] \text{ for } B \geq 7\} \cap$$
$$\{0 \leq B < 21, B \not\equiv 17\ [23] \text{ for } B \geq 17\}$$
$$\Leftrightarrow$$
$$B \in \{1,3,5,\cancel{7},9,11,13,15,\cancel{17},\cancel{19}\}$$
(here 7,17 and 19 are sieved, emphasized with a strike on numbers)

So left values of B: any odd number less than 21 and different that 7, 17 and 19

The statement of the proposition is confirmed using the values: a=21, n=1,2,3,4 and left B values:
Below is the numerical check for B=5:

- $\left(\frac{21+5}{2}+1=14\right)$ is not a multiple of $(6-1=5)$, $n=1$
- $\left(\frac{21+5}{2}+2=15\right)$ is not a multiple of $(12-1=11)$, $n=2$
- $\left(\frac{21+5}{2}+3=16\right)$ is not a multiple of $(18-1=17)$, $n=3$
- $\left(\frac{21+5}{2}+4=17\right)$ is not a multiple of $(24-1=23)$, $n=4$

The same can be verified all the B left values {1,3,5 (checked),9,11,13,15}

Proof of proposition 1.6:

Let be a, B and n natural numbers such that $a \geq 2$, $B<a$ and $n \geq 1$
On the first hand, , $\left(\frac{a+B}{2}\right)$ is a natural number only if B is of the same parity as a.

→ Thus, Constraint 1: B is of the same parity as a

On the second hand, by definition: $\left(\frac{a+B}{2}+n\right)$ is not a multiple of $(6n-1) \Leftrightarrow \left(\frac{a+B}{2}+n\right) \not\equiv 0\ [6n-1]$
$\Leftrightarrow \mathbf{B} \not\equiv -(\mathbf{a}+\mathbf{2n})\ [\mathbf{6n}-\mathbf{1}]$
So, the values of B satisfying $\left(\frac{a+B}{2}+n\right)$ *is not a multiple of* $(6n-1)$ are only B values that are non-congruent with $-(\mathbf{a}+\mathbf{2n})$ **modulo** $[\mathbf{6n}-\mathbf{1}]$: $\mathbf{B} \not\equiv -(\mathbf{a}+\mathbf{2n})\ [\mathbf{6n}-\mathbf{1}]$

→ Thus, Constraint 2: $\mathbf{B} \not\equiv -(\mathbf{a}+\mathbf{2n})\ [\mathbf{6n}-\mathbf{1}]$

On the third hand, to be more selective on Constraint 2, two cases need to be discussed:

- **Case 1:** $\left(\frac{\mathbf{a}+\mathbf{B}}{\mathbf{2}}+\mathbf{n}\right) < \mathbf{6n}-\mathbf{1}$**:**
  in this case $\left(\frac{a+B}{2}+n\right)$ can **never** be a multiple of $6n-1$
  Thus, in this case all natural numbers B such that $0 \leq B < a$ satisfy the expression
  $$\left(\frac{a+B}{2}+n\right) \textit{ is not a multiple of } (6n-1)$$
- **Case 2:** $\left(\frac{\mathbf{a}+\mathbf{B}}{\mathbf{2}}+\mathbf{n}\right) \geq \mathbf{6n}-\mathbf{1}$:
  in this case $\left(\frac{a+B}{2}+n\right) \geq 6n-1 \Leftrightarrow B \geq 2(5n-1)-a$

→ Scope of Constraint 2: The discussed 2 cases above resulting on targeting only B values:
$B \geq 2(5n-1)-a$ with the Constraint 2

Combining the constraints above to finely choose B such that:

- [Constraint 1]: B is of the same parity as a and
- [Constraint 2]: $\mathbf{B} \not\equiv -(\mathbf{a}+\mathbf{2n})\ [\mathbf{6n}-\mathbf{1}]$
- [Scope of Constraint 2]: For only constrained B values: $B \geq 2(5n-1)-a$

To sum it up using the sieve notation introduced previously to encapsulate the constraints 2:

- For evey B of the same parity as a:
- If $B \in \{0 \leq B < a, B \not\equiv -(a+2n)[6n-1]$ for $B \geq 2(5n-1)-a\}$
  Then $\left(\frac{a+B}{2}+n\right)$ **is not** a multiple of $(6n-1)$, for all numbers $n \geq 1$

And also, $\left(\frac{a+B}{2}+n\right) \geq 6n-1$ means that the upper limit of n is $\frac{2a+1}{10}$. Beyond this value, the assertion is trivial.

$$n \leq \frac{a+B+2}{10} \leq \frac{a+a-1+2}{10} = \frac{2a+1}{10} \quad \text{(because maximum n requires maximum B which is a-1)}$$

NB: The non-emptiness of the set $\{0 \leq B < a, B \not\equiv -(a+2n)[6n-1]$ for $B \geq 2(5n-1)-a\}$ is presumed

8. Important note:

Note that, the conditions related to the parameters B are cumulative across the previous 6 propositions in case we're seeking to satisfy combination of multiplicity assertions for a given "a". Subsequently more numbers are sieved from the initial range {0≤B<a} with respect to every sieve formula that comes in the combined assertions. NB: This paper presumes the non-emptiness of the set resulting from a sieve or combination of sieves. However, this assumption can be substantiated through the application of techniques used in sieve theory ([11]- Greaves, 2001).

For instance, in case of seeking for a combination where a B meets two conditions (multiplicity assertions):

$$\begin{cases} \left(\dfrac{a-B}{2} - n\right) \text{is not a multiple of } (6n-1) \\ \left(\dfrac{a+B}{2} - n\right) \text{is not a multiple of } (6n-1) \end{cases}$$

According to the propositions 1.1 and 1.2, the B must fulfill resulting conditions (on top of to $0 \leq B < a$):

$$\begin{cases} B \in \{0 \leq B < a, B \not\equiv (a-2n)\ [6n-1\ ] \text{ for } B \leq a - 2(7n-1)\} \\ B \in \{0 \leq B < a, B \not\equiv (2n-a)\ [6n-1\ ] \text{ for } B > 2(7n-1) - a\} \end{cases}$$

The final set of B values is the intersection of the $B - \text{Targets}$ sets of the proposition 1.1 and 1.2

→ **Example [10]**: for $a = 15$ and satisfying the combination above:

$$\begin{cases} B\ \in \{0 \leq B < 15, B\ \not\equiv 3\ [5\ ] \text{ for } B \leq 3\} \\ B\ \in \{0 \leq B < 15, B\ \not\equiv 2\ [5\ ] \text{ for } B \geq -3\} \\ B\ \in \{0 \leq B < 15, B \not\equiv 0\ [11\ ] \text{ for } B \geq 11\} \end{cases} \quad \Rightarrow \quad \text{Sieved values are struck: } B\ \in \{1,\cancel{3},5,\cancel{7},9,\cancel{11},13\}$$

Values of B : So, then any odd number less than 15 and different than 3, 7 and 11 is fitting: $B\ \in \{1,5,9,13\}$

VII. <u>Conjecture proof</u>

This approach relies on analyzing the structure of primes fitting in $6k \pm 1$ patterns. Subsequently, the primes 2 and 3 are considered as outliers to this approach since they're not fitting in any of $6k \pm 1$ patterns. 5 is indeed the (5= 6*1-1) is our starting prime number here.

Hence, the proof formally addresses even numbers $N \geq 10$. Specifically, the even numbers 4, 6, and 8 are considered outliers in this context due to their association with the specific primes 2 and 3, which do not adhere to any of the patterns $6k \pm 1$

In the proof, $N \geq 10$ is an even natural number and $A > 4$, a natural number such that $N = 2A$

As previously mentioned in the Structure of the even numbers section, in order to include all forms of A (and thereby all even numbers N), the three categories into which A can be classified separately will be addressed:

- Category 1: $A \equiv -1[3]$: A is of the form of "$3a - 1$"
- Category 2: $A \equiv +1[3]$: A is of the form of "$3a + 1$"
- Category 3: $A \equiv 0[3]$: A is of the form of "$3a$"

Each category is addressed comprehensively in three steps, drawing upon preceding propositions and corollaries.

1. Even natural numbers of Category 1: $\mathbf{A\ \equiv\ -1\ [3]}$

$$A\ \equiv\ -1\ [3] \Leftrightarrow \exists \text{ natural number } a\ \geq 2 \text{ such that } A = 3a - 1$$

- <u>**Step 1:** use of Propositions 1.1 and 1.2</u>

Let B be a natural number such that:

$$\mathbf{B - Conditions\ 1}: \begin{cases} 0 \leq B < a \\ \dfrac{a - B}{2} \text{ and } \dfrac{a + B}{2} \text{ are natural numbers} \\ \left(\dfrac{a - B}{2} - n\right) \text{is not a multiple of } (6n - 1) \\ \left(\dfrac{a + B}{2} - n\right) \text{is not a multiple of } (6n - 1) \end{cases}$$

Using the combination of the Propositions 1.1 and 1.2, the existence of a suitable B can be inferred: It suffices to select any natural number between 0 and $a - 1$ that shares the same parity as a and that fulfils following conditions:

$$\mathbf{B - Target\ 1}: \begin{cases} B \in \{0 \leq\ B < a, B\ \not\equiv (a - 2n)\ [6n - 1\ ] \text{ for } B \leq a - 2(7n - 1)\} \\ B \in \{0 \leq\ B < a, B\ \not\equiv (2n - a)\ [6n - 1\ ] \text{ for } B \geq 2(7n - 1) - a\} \end{cases}$$

- **Step 2:** use of Corollary-1

According to the Corollary-1 (using $B -$ Conditions 1), It is possible to assert that the coefficients

$k_p = \frac{a-B}{2}$ and $k_q = \frac{a+B}{2}$ are both 6k-1 prime generators.

And subsequently: $p = 6k_p - 1$ and $q = 6k_q - 1$ are primes.

In this context, the primes have been discerned and their form elucidated with respect to the structure of the even number N. Moreover, each eligible B yields a prime pair (p, q).

- **Step 3: Conclusion of Category 1**

Departing from the conventional structure of the even number N structure in this category, the composition of (p, q) pairs of primes in which it can be decomposed, is established.

The sum: $p + q = 6k_p - 1 + 6k_q - 1 = 6\left(\frac{a-B}{2} + \frac{a+B}{2}\right) - 2 = 6a - 2 = 2(3a - 1) = 2A = N$

This leads us to state that an even number N fitting in **category 1** can be written as the sum of two primes of the same pattern $6k - 1$

2. Even natural numbers Category 2: A $\equiv +1$ [3]

$$A\ \equiv\ +1\ [3] \Leftrightarrow \exists \text{ natural number } a\ \geq 2 \text{ such that } A = 3a + 1$$

- **Step 1:** use of Propositions 1.3, 1.4, 1.5, and 1.6:

Let be B a natural number such that:

$$\mathbf{Conditions\ 2}: \begin{cases} 0 \leq B < a \\ \dfrac{a - B}{2} \text{ and } \dfrac{a + B}{2} \text{ are natural numbers} \\ \left(\dfrac{a - B}{2} - n\right) \text{is not a multiple of } (6n + 1) \\ \left(\dfrac{a + B}{2} - n\right) \text{is not a multiple of } (6n + 1) \\ \left(\dfrac{a - B}{2} + n\right) \text{is not a multiple of } (6n - 1) \\ \left(\dfrac{a + B}{2} + n\right) \text{is not a multiple of } (6n - 1) \end{cases}$$

Using the combination of the Propositions 1.3, 1.4, 1.5, and 1.6, the existence of a suitable B can be inferred: It suffices to select any natural number between 0 and $a - 1$ that shares the same parity as a and that fulfils following conditions:

$$\mathbf{B-Target\ 2}: \begin{cases} B \in \{0 \leq B < a, B \not\equiv (a-2n)\ [6n+1]\ \text{for}\ B \leq a - 2(7n+1)\} \\ B \in \{0 \leq B < a, B \not\equiv (2n-a)\ [6n+1]\ \text{for}\ B \geq 2(7n+1) - a\} \\ B \in \{0 \leq B < a, B \not\equiv (a+2n)\ [6n-1]\ \text{for}\ B \leq a - 2(5n-1)\} \\ B \in \{0 \leq B < a, B \not\equiv -(a+2n)\ [6n-1]\ \text{for}\ B \geq 2(5n-1) - a\} \end{cases}$$

- **Step 2:** use of Corollary 2

According to the Corollary 2 (using $B-\text{Conditions}\ 2$ above), It is possible to assert that the coefficients:

$k_p = \frac{a-B}{2}$ and $k_q = \frac{a+B}{2}$ are both $6k + 1$ prime generators.

And subsequently, $p = 6k_p + 1$ and $q = 6k_q + 1$ are primes

In this context, the primes have been discerned and their form elucidated with respect to the structure of the even number N. Moreover, each eligible B yields a prime pair (p, q).

- **Step 3: Conclusion of category 2**

Departing from the conventional of the even number N structure in this category, the composition of (p, q) pairs of primes in which it can be decomposed, is established.

The sum: $p + q = 6k_p + 1 + 6k_q + 1 = 6\left(\frac{a-B}{2} + \frac{a+B}{2}\right) + 2 = 6a + 2 = 2(3a + 1) = 2A = N$

This leads us to state that an even number N fitting in **category 2** can be written as the sum of two primes having the same pattern $6k + 1$

3. Even natural numbers Category 3: $A \equiv 0\ [3]$

$$A \equiv 0\ [3] \Leftrightarrow \exists \text{ natural number } a \geq 2 \text{ such that } A = 3a$$

In this category, there are two sieve combinations that can be made. We explore these 2 options below:

i) **Category 3 - Option 1**

- **Step 1:** use of Propositions 1.1, 1.4 and 1.6

Let be B a natural number natural number such that:

$$\mathbf{B-Conditions\ 3.1}: \begin{cases} 0 \leq B < a \\ \frac{a-B}{2} \text{ and } \frac{a+B}{2} \text{ are natural numbers} \\ \left(\frac{a-B}{2} - n\right) \text{is not a multiple of } (6n-1) \\ \left(\frac{a+B}{2} - n\right) \text{is not a multiple of } (6n+1) \\ \left(\frac{a+B}{2} + n\right) \text{is not a multiple of } (6n-1) \end{cases}$$

Using to the Proposition 1.1, 1.4 and 1.6, the existence of a suitable B can be inferred: It suffices to select any natural number between 0 and $a - 1$ that shares the same parity as a and that fulfils following conditions:

$$\mathbf{B-Target\ 3.1}: \begin{cases} B \in \{0 \leq B < a, B \not\equiv (a-2n)\ [6n-1\ ] \text{ for } B \leq a - 2(7n-1)\} \\ B \in \{0 \leq B < a, B \not\equiv (2n-a)\ [6n+1\ ] \text{ for } B \geq 2(7n+1) - a\} \\ B \in \{0 \leq B < a, B \not\equiv -(a+2n)\ [6n-1\ ] \text{ for } B \geq 2(5n-1) - a\} \end{cases}$$

- **Step 2:** use of Corollaries 1 and 2

According to the Corollaries 1 and 2 (using $B-$ Conditions 3.1 above), It is possible to assert that:

- The coefficient $k_p = \frac{a-B}{2}$ is $6k-1$ prime generator and
- The coefficient $k_q = \frac{a+B}{2}$ is $6k+1$ prime generator

And subsequently, $p = 6k_p - 1$ and $q = 6k_q + 1$ are primes

In this context, the primes have been discerned and their form elucidated with respect to the structure of the even number N. Moreover, each eligible B yields a prime pair (p, q).

- **Step 3: conclusion of Category 3 - Option 1**

Departing from the conventional structure of the even number N in this option of the category 3, the composition of (p, q) pairs of primes in which it can be decomposed, is established.

The sum: $p + q = 6k_p - 1 + 6k_q + 1 = 6\left(\frac{a-B}{2} + \frac{a+B}{2}\right) = 6a = 2(3a) = 2A = N$

This leads us to state that an even number N fitting in this **first option of the category** 3, can be written as the sum of two primes having different patterns: the smaller prime is $6k-1$ like and the bigger prime is $6k+1$ like.

ii) **Category 3 - Option 2**

- **Step 1:** use of Propositions 1.3, 1.2 and 1.5

Let be B a natural number natural number such that:

$$\mathbf{B-Conditions\ 3.2}: \begin{cases} 0 \leq B < a \\ \frac{a-B}{2} \text{ and } \frac{a+B}{2} \text{ are natural numbers} \\ \left(\frac{a-B}{2} - n\right) \text{is not a multiple of } (6n+1) \\ \left(\frac{a+B}{2} - n\right) \text{is not a multiple of } (6n-1) \\ \left(\frac{a-B}{2} + n\right) \text{is not a multiple of } (6n-1) \end{cases}$$

Using to the Propositions 1.3, 1.2 and 1.5, the existence of a suitable B can be inferred: It suffices to select any natural number between 0 and $a-1$ that shares the same parity as a and that fulfils following conditions:

$$\mathbf{B-Target\ 3.2}: \begin{cases} B \in \{0 \leq B < a, B \not\equiv (a-2n)\ [6n+1\ ] \text{ for } B \leq a - 2(7n+1)\} \\ B \in \{0 \leq B < a, B \not\equiv (2n-a)\ [6n-1\ ] \text{ for } B \geq 2(7n-1) - a\} \\ B \in \{0 \leq B < a, B \not\equiv (a+2n)\ [6n-1\ ] \text{ for } B \leq a - 2(5n-1)\} \end{cases}$$

- **Step 2:** use of Corollaries 1 and 2

According to the Corollaries 1 and 2 (using $B-$ Conditions 3.2 above), It is possible to assert that:

- The coefficient $k_p = \frac{a-B}{2}$ is $6k+1$ prime generator and
- The coefficient $k_q = \frac{a+B}{2}$ is $6k-1$ prime generator

And subsequently, $p = 6k_p + 1$ and $q = 6k_q - 1$ are primes.

In this context, the primes have been discerned and their form elucidated with respect to the structure of the even number N. Moreover, each eligible B yields a prime pair (p, q).

- **Step 3: conclusion of Category 3 - Option 2**

Departing from the conventional structure of the even number N in this option of the category 3, the composition of (p, q) pairs of primes in which it can be decomposed, is established.

The sum: $p + q = 6k_p + 1 + 6k_q - 1 = 6\left(\frac{a-B}{2} + \frac{a+B}{2}\right) = 6a = 2(3a) = 2A = N$

This leads us to state that an even number N fitting in this **second option of the category 3**, can be written as the sum of two primes having different patterns: the smaller prime is $6k + 1$ like and the bigger prime is $6k - 1$ like.

To sum this up, by considering the three categories, all even natural numbers ≥ 10 are encompassed.

In summary, depending on $\frac{N}{2}$ shape ($\frac{N}{2} = A$), any even number $N \geq 10$ can be expressed as the sum of a pair of prime numbers in the following manner:

→ Exclusively, of pair of $6k - 1$ primes like, if $\frac{N}{2}$ is of the $3a - 1$ shape

→ Exclusively, of pair of $6k + 1$ primes like, if $\frac{N}{2}$ is of the $3a + 1$ shape

→ by the combination of both $6k + 1$ like and $6k - 1$ like, if $\frac{N}{2}$ is of the $3a$ shape. As previously discussed, (in the Category 3), the order of combination in the latter case is relevant (Option 1 Vs Option 2).

In the section below random examples to illustrate the proof logic for each of the discussed categories above.

VIII. Categories illustrations with real examples

In this section, random examples will be examined to illustrate the three categories discussed above in which can fall any even natural number and the reasoning presented in the proof. Indeed, as presented early in the proof, a given even number $N (= 2A)$ falls into one of three categories inherited from the natural number A (or $\frac{N}{2}$): $(3a - 1)$ category or $(3a + 1)$ category or $(3a)$ category. Every category will be addressed with its dedicated example by identifying the pairs of primes (p, q) surrounding the natural number A and thus lead to N: $p + q = 2A = N$.

The mechanic used involves identifying B parameters that lead to prime-generators pairs (k_p, k_q) which subsequently lead to the (p, q) pair of primes surrounding the natural number A.

As a reminder: $p = 6k_p \pm 1$, $q = 6k_q \pm 1$, $k_p = \frac{a-B}{2}$, $k_q = \frac{a+B}{2}$, $A = 3a \pm 1$ or $A = 3a$, $p + q = 2A = N$

1. Example [11] for Category 1: $A \equiv -1$ [3]: A= 44 (44 = 3 * 15 – 1 so: a = 15 and N= 88)

As per our findings, in this category the (p, q) pairs of primes are exclusively $6k - 1$ like

And the B parameters have to be odd (same parity as a) and must satisfy the $B - \text{Target } 1$ system:

$$B - \text{Target } 1: \begin{cases} B \in \{0 \leq B < a, B \not\equiv (a - 2n)\ [6n - 1\ ] \text{ for } B \leq a - 2(7n - 1)\}; \text{range of n: } 1 \leq n \leq \dfrac{a + 2}{14} \\ B \in \{0 \leq B < a, B \not\equiv (2n - a)\ [6n - 1\ ] \text{ for } B \geq 2(7n - 1) - a\}; \text{range of n: } 1 \leq n \leq \dfrac{2a + 1}{14} \end{cases}$$

This means after replacing with the values:

$$B - \text{Target } 1: \begin{cases} B \in \{0 \leq B < 15, B \not\equiv 3[5]; B \leq 3\} \\ B \in \{0 \leq B < 15, B \not\equiv 2[5]; B \geq -3\} \cap \{0 \leq B < 15, B \not\equiv 0[11]; B \geq 11\} \end{cases} \left| \begin{matrix} n = 1 \\ n = 1,2 \end{matrix} \right\}$$

By applying the sieve mechanic for identifying the not sieved values (struck values are the one to be excluded)

$$\text{B} - \text{Target 1: } \begin{cases} B \in \{1,\cancel{3},5,7,9,11,13\} \\ B \in \{1,3,5,\cancel{7},9,\cancel{11},13\} \end{cases} \left| \begin{matrix} n = 1 \\ n = 1,2 \end{matrix} \right\} \Longrightarrow B \in \{1,\cancel{3},5,\cancel{7},9,\cancel{11},13\}$$

Finally, the resulting values of B are: $\{\mathbf{1},\mathbf{5},\mathbf{9},\mathbf{13}\}$

So, then any odd number less than 15 (a=15) and different than 3, 7 and 11 must be fitting. Here is a numerical check with all the final resulting B values highlighting related prime generators and pair of primes surrounding A for this example:

→ For B = 1: $k_p = \frac{a-B}{2} = \frac{15-1}{2} = 7$ and $k_q = \frac{a+B}{2} = \frac{15+1}{2} = 8$ generate p= 6*7-1 = 41 and q=6*8-1 = 47

→ For B= 5: $k_p = \frac{a-B}{2} = \frac{15-5}{2} = 5$ and $k_q = \frac{a+B}{2} = \frac{15+5}{2} = 10$ generate p= 6*5-1 = 29 and q=6*10-1 = 59

→ For B= 9: $k_p = \frac{a-B}{2} = \frac{15-9}{2} = 3$ and $k_q = \frac{a+B}{2} = \frac{15+9}{2} = 12$ generate p= 6*3-1 = 17 and q=6*12-1 = 71

→ For B= 13: $k_p = \frac{a-B}{2} = \frac{15-1}{2} = 1$ and $k_q = \frac{a+B}{2} = \frac{15+13}{2} = 14$ generate p= 6*1-1 = 5 and q=6*8-1 = 83

And here is the numerical check for the sieved B values 3, 7 and 11that are expected not to generate pair of primes:

× For B = 3: $k_p = \frac{a-B}{2} = \frac{15-3}{2} = 6$ and $k_q = \frac{a+B}{2} = \frac{15+3}{2} = 9$ generate p= 6*6-1 = 35 and q=6*9-1 = 53

× For B= 7: $k_p = \frac{a-B}{2} = \frac{15-7}{2} = 4$ and $k_q = \frac{a+B}{2} = \frac{15+7}{2} = 11$ generate p= 6*4-1 = 23 and q=6*11-1 = 65

× For B= 11: $k_p = \frac{a-B}{2} = \frac{15-11}{2} = 2$ and $k_q = \frac{a+B}{2} = \frac{15+11}{2} = 13$ generate p= 6*2-1 = 11 and q=6*13-1 = 77

2. Example [12] for Category 2: A ≡ +1 [3]: A= 55 (55 = 3 * 18 +1 so: a = 18 and N= 110)

As per our findings, in this category the (p, q) pairs of primes are exclusively $6k + 1$ like

And the B parameters have to be even (same parity as a) and must satisfy the $B - \text{Target 2}$ system:

$$B - \text{Target 2: } \begin{cases} B \in \{0 \leq B < a, B \not\equiv (a - 2n)\ [6n+1] \text{ for } B \leq a - 2(7n+1)\}; \text{range of n: } 1 \leq n \leq \dfrac{a-2}{14} \\ B \in \{0 \leq B < a, B \not\equiv (2n - a)\ [6n+1] \text{ for } B \geq 2(7n+1) - a\}; \text{range of n: } 1 \leq n \leq \dfrac{2a-2}{14} \\ B \in \{0 \leq B < a, B \not\equiv (a + 2n)\ [6n-1] \text{ for } B \leq a - 2(5n-1)\}; \text{; range of n: } 1 \leq n \leq \dfrac{a+2}{10} \\ B \in \{0 \leq B < a, B \not\equiv -(a + 2n)\ [6n-1] \text{ for } B \geq 2(5n-1) - a\}; \text{; range of n: } 1 \leq n \leq \dfrac{2a+1}{10} \end{cases}$$

This means after replacing with the values:

$$B - \text{Target 2} \begin{cases} B \in \{0 \leq B < 18, B \not\equiv 2[7] \text{for } B \leq 2\} \\ B \in \{0 \leq B < 18, B \not\equiv 5[7] \text{for} B \geq -2\} \cap \{0 \leq B < 18, B \not\equiv 12[13] \text{for } B \geq 12\} \\ B \in \{0 \leq B < 18, B \not\equiv 0[5] \text{for } B \leq 10\} \\ B \in \{0 \leq B < 18, B \not\equiv 0[5] \text{ for } B \geq -10\} \cap \{0 \leq B < 18, B \not\equiv 0[11] \text{for } B \geq 0\} \\ \cap \{0 \leq B < 18, B \not\equiv 0[17] \text{for } B \geq 10\} \end{cases} \left| \begin{matrix} n = 1 \\ n = 1,2 \\ n = 1 \\ n = 1,2,3 \end{matrix} \right\}$$

By applying the sieve mechanic for identifying the not sieved values (struck values are the one to be excluded)

$$B - \text{Target 2} \begin{cases} B \in \{0,\cancel{2},4,6,8,10,12,14,16\} \\ B \in \{0,2,4,6,8,10,\cancel{12},14,16\} \cap B \in \{0,2,4,6,8,10,\cancel{12},14,16\} \\ B \in \{\cancel{0},2,4,6,8,\cancel{10},12,14,16\} \\ B \in \{\cancel{0},2,4,6,8,\cancel{10},\cancel{12},14,16\} \cap \{\cancel{0},2,4,6,8,10,\cancel{12},14,16\} \\ \cap \{\cancel{0},2,4,6,8,10,12,14,16\} \end{cases} \left| \begin{matrix} n = 1 \\ n = 1,2 \\ n = 1 \\ n = 1,2,3 \end{matrix} \right\}$$

This leads to: $B - \text{Target } 2: B \in \{\cancel{0}, \cancel{2}, 4, 6, 8, \cancel{10}, \cancel{12}, 14, 16\}$

Finally, the resulting values of B are: **{4, 6, 8, 14, 16}**

So, then any even number less than 18 and different than 0, 2, 10, and 12 must be fitting. Here is a numerical check with all the final resulting B values highlighting related prime generators and pair of primes surrounding A for this example:

→ For B = 4: $k_p = \frac{a-B}{2} = \frac{18-4}{2} = 7$ and $k_q = \frac{a+B}{2} = \frac{18+4}{2} = 11$ generate p= 6*7+1 = 43 and q=6*11+1 = 67

→ For B= 6: $k_p = \frac{a-B}{2} = \frac{18-6}{2} = 6$ and $k_q = \frac{a+B}{2} = \frac{18+6}{2} = 12$ generate p= 6*6+1 = 37 and q=6*12+1 = 73

→ For B= 8: $k_p = \frac{a-B}{2} = \frac{18-8}{2} = 5$ and $k_q = \frac{a+B}{2} = \frac{18+8}{2} = 13$ generate p= 6*5+1 = 31 and q=6*13+1 = 79

→ For B= 14: $k_p = \frac{a-B}{2} = \frac{18-14}{2} = 2$ and $k_q = \frac{a+B}{2} = \frac{18+14}{2} = 16$ generate p= 6*2+1 = 13 and q=6*16+1 = 97

→ For B= 16: $k_p = \frac{a-B}{2} = \frac{18-16}{2} = 1$ and $k_q = \frac{a+B}{2} = \frac{18+16}{2} = 17$ generate p= 6*1+1 = 7 and q=6*17+1 = 103

And here is the numerical check for the sieved B values 0, 2, 10, and 12 that are expected not to generate pair of primes:

× For B = 0: $k_p = \frac{a-B}{2} = \frac{18-0}{2} = 9$ and $k_q = \frac{a+B}{2} = \frac{18+0}{2} = 9$ generate p= 6*9+1 = 55 and q=6*9+1 = 55

× For B= 2: $k_p = \frac{a-B}{2} = \frac{18-2}{2} = 8$ and $k_q = \frac{a+B}{2} = \frac{18+2}{2} = 10$ generate p= 6*8+1 = 49 and q=6*10+1 = 61

× For B= 10: $k_p = \frac{a-B}{2} = \frac{18-10}{2} = 4$ and $k_q = \frac{a+B}{2} = \frac{18+10}{2} = 14$ generate p= 6*4+1 = 25 and q=6*14+1 = 85

× For B= 12: $k_p = \frac{a-B}{2} = \frac{18-12}{2} = 3$ and $k_q = \frac{a+B}{2} = \frac{18+12}{2} = 15$ generate p= 6*3+1 = 19 and q=6*15+1 = 91

3. Example [13] for Category 3 Option 1: A ≡ 0 [3]: A= 51 (51 = 3 * 17 so: a = 17 and N= 102)

As per our findings, the first option in this category: p is $6k - 1$ like and q is $6k + 1$ like

And the B parameters have to be odd (same parity as a) and must satisfy the $B - \text{Target } 3.1$ system:

$$B - \text{Target } 3.1: \begin{cases} B \in \{0 \le B < a, B \not\equiv (a-2n)\ [6n-1]\ \text{for } B \le a - 2(7n-1)\}; \text{range of n: } 1 \le n \le \dfrac{a+2}{14} \\ B \in \{0 \le B < a, B \not\equiv (2n-a)\ [6n+1]\ \text{for } B \ge 2(7n+1) - a\}; \text{range of n: } 1 \le n \le \dfrac{2a-3}{14} \\ B \in \{0 \le B < a, B \not\equiv -(a+2n)\ [6n-1]\ \text{for } B \ge 2(5n-1) - a\};\ ; \text{range of n: } 1 \le n \le \dfrac{2a+1}{10} \end{cases}$$

This means after replacing with the values:

$$B - \text{Target } 3.1: \left\{ \begin{array}{c|c} B \in \{0 \le B < 17, B \not\equiv 0[5] \text{for } B \le 5\} & n = 1 \\ B \in \{0 \le B < 17, B \not\equiv 6[7] \text{ for } B \ge -1\} \cap \{0 \le B < 17, B \not\equiv 0[13] \text{ for } B \ge 13\} & n = 1,2 \\ B \in \{0 \le B < 17, B \not\equiv 1[5] \text{ for } B \ge -9\} \cap \{0 \le B < 17, B \not\equiv 1[11] \text{ for } B \ge 1\} & n = 1,2 \end{array} \right\}$$

By applying the sieve mechanic for identifying the not sieved values (struck values are the one to be excluded):

$$B - \text{Target } 3.1: \left\{ \begin{array}{c|c} B \in \{1, 3, \cancel{5}, 7, 9, 11, 13, 15\} & n = 1 \\ B \in \{1, 3, 5, 7, 9, 11, \cancel{13}, 15\} & n = 1,2 \\ B \in \{\cancel{1}, 3, 5, 7, 9, \cancel{11}, 13, 15\} & n = 1,2 \end{array} \right\}$$

Finally, the resulting values of B are: **{3, 7, 9, 15}**

So, then any odd number less than 17 and different than 1, 5, 11 and 13 must be fitting. Here is a numerical check with all the final resulting B values highlighting related prime generators and pair of primes surrounding A for this example:

→ B = 3:$k_p = \frac{a-B}{2} = \frac{17-3}{2} = 7$ and $k_q = \frac{a+B}{2} = \frac{17+3}{2} = 10$ generate p= 6*7-1 = 41 and q=6*10+1 = 61

→ B= 7: $k_p = \frac{a-B}{2} = \frac{17-7}{2} = 5$ and $k_q = \frac{a+B}{2} = \frac{17+7}{2} = 12$ generate p= 6*5-1 = 29 and q=6*12+1 = 73

→ B= 9: $k_p = \frac{a-B}{2} = \frac{17-9}{2} = 4$ and $k_q = \frac{a+B}{2} = \frac{17+9}{2} = 13$ generate p= 6*4-1 = 23 and q=6*13+1 = 79

→ B= 15: $k_p = \frac{a-B}{2} = \frac{17-15}{2} = 1$ and $k_q = \frac{a+B}{2} = \frac{17+1}{2} = 16$ generate p= 6*1-1 = 5 and q=6*16+1 = 97

And here is the numerical check for the sieved B values 1, 5, 11 and 13 that are expected not to generate pair of primes:

× B = 1:$k_p = \frac{a-B}{2} = \frac{17-1}{2} = 8$ and $k_q = \frac{a+B}{2} = \frac{17+1}{2} = 9$ generate p= 6*8-1 = 47 and q=6*9+1 = 55

× B= 5: $k_p = \frac{a-B}{2} = \frac{17-5}{2} = 6$ and $k_q = \frac{a+B}{2} = \frac{17+5}{2} = 11$ generate p= 6*6-1 = 35 and q=6*11+1 = 67

× B= 11: $k_p = \frac{a-B}{2} = \frac{17-11}{2} = 3$ and $k_q = \frac{a+B}{2} = \frac{17+11}{2} = 14$ generate p= 6*3-1 = 17 and q=6*14+1 = 85

× B= 13: $k_p = \frac{a-B}{2} = \frac{17-1}{2} = 2$ and $k_q = \frac{a+B}{2} = \frac{17+13}{2} = 15$ generate p= 6*2-1 = 11 and q=6*15+1 = 91

4. Example [14] for Category 3 Option 2: A ≡ 0 [3]: A= 51 (51 = 3 * 17 so: a = 17 and N= 102)

As per our findings, the second option in this category: p is $6k + 1$ like and q is $6k - 1$ like

And the B parameters have to be odd (same parity as a) and must satisfy the $B - \text{Target } 3.2$ system

$$B - \text{Target } 3.2: \begin{cases} B \in \{0 \le B < a, B \not\equiv (a - 2n)\ [6n + 1\ ] \text{ for } B \le a - 2(7n + 1)\}; \text{range of n: } 1 \le n \le \dfrac{a-2}{14} \\ B \in \{0 \le B < a, B \not\equiv (2n - a)\ [6n - 1\ ] \text{ for } B > 2(7n - 1) - a\}; \text{range of n: } 1 \le n \le \dfrac{2a+1}{14} \\ B \in \{0 \le B < a, B \not\equiv (a + 2n)\ [6n - 1\ ] \text{ for } B \le a - 2(5n - 1)\};\ ; \text{range of n: } 1 \le n \le \dfrac{a+2}{10} \end{cases}$$

This means after replacing with the values:

$$B - \text{Target } 3.2: \left\{ \begin{array}{c} B \in \{\ 0 \le B < 17, B\ \not\equiv 1[7] \text{ for } B \le 1\} \\ B \in \{\ 0 \le B < 17, B\ \not\equiv 0[5] \text{ for } B \ge -5\} \cap \{0 \le B < 17, B\ \not\equiv 9[11] \text{ for } B \ge 9\} \\ B \in \{0 \le B < 17, B\ \not\equiv 4[5] \text{ for } B \le 9\} \end{array} \middle| \begin{array}{c} n = 1 \\ n = 1,2 \\ n = 1 \end{array} \right\}$$

By applying the sieve mechanic for identifying the not sieved values (struck values are the one to be excluded):

$$B - \text{Target } 3.2: \left\{ \begin{array}{c} B \in \{\cancel{1}, 3, 5, 7, 9, 11, 13, 15\} \\ B \in \{1, 3, \cancel{5}, 7, \cancel{9}, 11, 13, \cancel{15}\} \\ B \in \{1, 3, 5, 7, \cancel{9}\} \end{array} \middle| \begin{array}{c} n = 1 \\ n = 1,2 \\ n = 1 \end{array} \right\}$$

Finally, the resulting values of B are: **{3, 7, 11, 13}**

So, then any odd number less than 17 and different than 1, 5, 9 and 15 must be fitting. Here is a numerical check with all the final resulting B values highlighting related prime generators and pair of primes surrounding A for this example:

→ B = 3: $k_p = \frac{a-B}{2} = \frac{17-3}{2} = 7$ and $k_q = \frac{a+B}{2} = \frac{17+3}{2} = 10$ generates p= 6*7+1 = 43 and q=6*10-1 = 59

→ B= 7: $k_p = \frac{a-B}{2} = \frac{17-7}{2} = 5$ and $k_q = \frac{a+B}{2} = \frac{17+7}{2} = 12$ generates p= 6*5+1 = 31 and q=6*12-1 = 71

→ B= 11: $k_p = \frac{a-B}{2} = \frac{17-11}{2} = 3$ and $k_q = \frac{a+B}{2} = \frac{17+11}{2} = 14$ generates p= 6*3+1 = 19 and q=6*14+-1 = 83

→ B= 13: $k_p = \frac{a-B}{2} = \frac{17-13}{2} = 2$ and $k_q = \frac{a+B}{2} = \frac{17+13}{2} = 15$ generates p= 6*2+1 = 13 and q=6*15-1 = 89

And here is the numerical check for the sieved B values 1, 5, 9 and 15 that are expected not to generate pair of primes

× B = 1: $k_p = \frac{a-B}{2} = \frac{17-1}{2} = 8$ and $k_q = \frac{a+B}{2} = \frac{17+1}{2} = 9$ generates p= 6*8+1 = 49 and q=6*9-1 = 53

× B = 5: $k_p = \frac{a-B}{2} = \frac{17-5}{2} = 6$ and $k_q = \frac{a+B}{2} = \frac{17+5}{2} = 11$ generates p= 6*6+1 = 37 and q=6*11-1 = 65

× B = 9: $k_p = \frac{a-B}{2} = \frac{17-9}{2} = 4$ and $k_q = \frac{a+B}{2} = \frac{17+9}{2} = 13$ generates p= 6*4+1 = 25 and q=6*13-1 = 77

× B = 15: $k_p = \frac{a-B}{2} = \frac{17-15}{2} = 1$ and $k_q = \frac{a+B}{2} = \frac{17+15}{2} = 16$ generates p= 6*1+1 = 7 and q=6*16-1 = 95

In all the examples provided above covering all types of numbers, both the sieved and unsieved B parameters have been verified and behave as expected.

IX. Conclusion

Even numbers have been grouped into three categories based on their structures and their relationship to multiples of 3. Each category has been thoroughly explained, detailing the specific form and describing the primes involved, utilizing the structural factor "**a**" and the prime generator parameter "**B**" finely selected using the ad-hoc sieve notation introduced in this paper. A formulation has been derived to describe all pair of primes within each category of even numbers yielded by eligible "**B**" parameters. The primes relevant to this approach are those adhering to the "**6k±1**" patterns (primes greater than or equal to **5**). In this context, emphasis is placed on natural even numbers greater than or equal to **10**. Even numbers below **10** are treated as special cases or exceptions due to their involvement with the specific prime numbers **2** and **3.**

This elucidation allows for the decomposition of any given even number greater or equal to **10**. This intends to provide an elementary demonstration for the *Strong Goldbach's* conjecture for all even numbers greater or equal to **10.**

X. Direct applications of the approach

1. Direct application for twin primes identification:

As a direct application of this proof, it becomes apparent that twin primes exclusively encircle numbers of the pattern $A \equiv 0\ [3]$.

Indeed, by definition, primes are odd, and every pair of twin primes are separated by a distance of 2.

So, the lower prime must be of $6k - 1$ like and the bigger must be $6k + 1$ like, it cannot be otherwise. According to our proof, this on possible in the Category 3 – Option 1.

$q - p = (6k_q + 1) - (6k_p - 1) = 2$ implies $k_q = k_p = \frac{A}{6}$ (A needs to be multiple of 6)

$k_p = k_q$ implies $\frac{a-B}{2} = \frac{a+B}{2}$ implies $B = 0$. Simply means that **0 needs to be part of the solutions of the** $B -$ Target 3.1:

In summary:

- A needs to be multiple of 6 and
- **0 needs to be part of the solutions of the** $B -$ Target 3.1:

$$\mathbf{B - Target\ 3.1}: \begin{cases} B \in \{0 \le B < a, B \not\equiv (a - 2n)\ [6n - 1\,]\ \text{for}\ B \le a - 2(7n - 1)\} \\ B \in \{0 \le B < a, B \not\equiv (2n - a)\ [6n + 1\,]\ \text{for}\ B \ge 2(7n + 1) - a\} \\ B \in \{0 \le B < a, B \not\equiv -(a + 2n)\ [6n - 1\,]\ \text{for}\ B \ge 2(5n - 1) - a\} \end{cases}$$

By replacing in the system $B = 0$ and $a = \frac{A}{3}$: there are 4 conditions need to be met:

1. A multiple of 6 and
2. $\left\{0 \not\equiv \left(\frac{A}{3} - 2n\right)\ [6n - 1\,]\ \text{for}\ n \le \frac{\frac{A}{6}+1}{7}\right\}$ and
3. $\left\{0 \not\equiv \left(2n - \frac{A}{3}\right)\ [6n + 1\,]\ \text{for}\ \ n \le \frac{\frac{A}{6}-1}{7}\right\}$ and
4. $\left\{0 \not\equiv -\left(\frac{A}{3} + 2n\right)\ [6n - 1\,]\ \text{for}\ 0\ n \le \frac{\frac{A}{6}+1}{5}\right\}$

For a given number, checking that it's encircled by twin primes comes to check these 4 conditions above.

→ **Example [15]:**

Here's an illustration showcasing this mechanism, employing different natural numbers A, all of which are multiples of 6.

The table displays the second, third, and fourth conditions that must be satisfied (with the first condition, being a multiple of 6, already fulfilled) along with all iterations of 'n' that need to be calculated according to the formulas. The final column serves as a countercheck of the actual twin primes reality.

As evident, there exists a perfect alignment between the fulfillment of the four conditions and the verification of the values.

A	$\left(\frac{A}{3} - 2n\right)\ [6n - 1\,]$; $n \le \frac{\frac{A}{6}+1}{7}$	$\left(2n - \frac{A}{3}\right)\ [6n + 1\,]$;$n \le \frac{\frac{A}{6}-1}{7}$	$-\left(\frac{A}{3} + 2n\right)\ [6n - 1\,]$; $n \le \frac{\frac{A}{6}+1}{5}$	4 Conditions checking as per above	Verification whether A is surrounded by twin primes
30	3 $n \le 1$	6 $n \le 1$	3 $n \le 1$	Yes	Yes (29,31)
36	0 $n \le 1$	4 $n \le 1$	1 $n \le 1$	No (there is a 0)	No (~~35~~,37)
42	2 $n \le 1$	2 $n \le 1$	4 $n \le 1$	Yes	Yes (41,43)
60	3 $n \le 1$	3 $n \le 1$	3 $n \le 1$	Yes	Yes (59,61)
120	3,3,0 $n \le 3$	3,3,4 $n \le 1$	3,0,5,21 $n \le 4$	No (there is a 0)	No (119,~~121~~)
600	3,9,7,8,16,13,22,43 $n \le 8$	5,12,15,8,27,34,29,12 $n \le 8$	3,5,15,22,22,33,32,19,47 $n \le 12$	Yes	Yes (599,601)

2. Direct application for a new Primality test:

For a given natural number A to be a prime, it must only fall in categories 1 or 2: $A \equiv \pm 1[3]$ and p, q and A must coincide. This implies $B = 0$ **must be a solution of the** $B - \text{Target } 1$ or $B - \text{Target } 2$ depending on the structure of A.

By replacing in the system $B = 0$ and $a = \frac{A \pm 1}{3}$ in the systems and proceeding to some simplifications in the notation:

- **Category 1**: $A \equiv -1[3]$: $\mathbf{B - Target\ 1}$: $\left\{ 0 \not\equiv \pm\left(\frac{A+1}{3} - 2n\right) [6n-1] \text{ for } n \leq \frac{\left(\frac{A+1}{6}+1\right)}{7} \right\}$

- **Category 2**: $A \equiv +1[3]$: $\mathbf{B - Target\ 2}$: $\begin{cases} 0 \not\equiv \pm\left(\frac{A-1}{3} - 2n\right) [6n+1] \text{ for } n \leq \frac{\left(\frac{A-1}{6}-1\right)}{7} \\ 0 \not\equiv \pm\left(\frac{A-1}{3} + 2n\right) [6n-1] \text{ for } n \leq \frac{\left(\frac{A-1}{6}+1\right)}{5} \end{cases}$

In other words, when determining if a natural number A (depending on its category) is prime, you only need to verify that the remainders of its Euclidean divisions mentioned above aren't equal to 0.

3. Direct application for Identification of series of primes:

Taking another viewpoint into account, when A is a prime number of any kind ($6k \pm 1$), and considering the solutions of the B parameters in the respective B-TARGET 1 & 2 systems mentioned earlier, distinctive sequences of prime numbers that have differences of exactly 3B can be identified:

$$p = 6k_p \pm 1 = 6\frac{a-B}{2} \pm 1 = 3a \pm 1 - 3B = \mathbf{A - 3B} \text{ and } q = 6k_q \pm 1 = 6\frac{a+B}{2} \pm 1 = 3a \pm 1 + 3B = \mathbf{A + 3B}$$

The triplet $(\mathbf{A - 3B, A, A + 3B})$ **is a triplet of primes.**

For instance, in case of B=2 serves as a solution in the relevant B-TARGET system, it signifies that A forms triplet of "sexy primes[2]" (where A-6, A and A+6 are primes). This pattern continues depending on the discussed values of B. The smaller B is, the closer the primes are to each other in the sequence (according to the distance 3B between them).

$$p = 6k_p \pm 1 = 6\frac{a-2}{2} \pm 1 = 3a \pm 1 - 6 = A - 6 \text{ and } q = 6k_q \pm 1 = 6\frac{a+2}{2} \pm 1 = 3a \pm 1 + 6 = A + 6$$

→ **Example [16]:** Example for Category 1: for the prime number $A = 53$ $(A = 3 * 18 - 1)$:
Employing the method outlined in the Proof illustration section, the final parameter B values that are not sieved for this example are : {0,**2**,10,12, 16}
So $\mathbf{B} = 2$ being a solution to the $B - \text{Target } 1$. Subsequently $(A - 6, A, A + 6)$ must be a triplet of sexy prime:
Numerical check: $(\mathbf{47, 53, 59})$ is indeed a triplet of sexy primes

→ **Example [17]:** Example for Category 2: for the prime number $A = 43$ $(A = 3 * 14 + 1)$:
Employing the method outlined in the Proof illustration section, the final parameter B values that are not sieved for this example are : {0,**2**,8,10,12}
So $B = 2$ being a solution to the $B - \text{Target } 2$: Subsequently $(A - 6, A, A + 6)$ must be a triplet of sexy prime. Numerically: $(\mathbf{37, 43, 49})$ is indeed a triplet of sexy primes.

4. Direct application for Goldbach 's Comet:

When plotting the count of all the un-sieved **"B"** parameters for each natural even number, it results in drawing the so called Goldbach's comet ([6]- Donato Saeli, 2012).

[2] Sexy primes are primes that have a difference of 6, for example: (5, 11), (7, 13), (11, 17)

XI. Références

- [1]- Tattersall, J. J. (1999). *Elementary Number Theory in Nine Chapters - ISBN 978-0-521-58503-3.* Cambridge University Press.
- [2]- Wikipédia. (n.d.). *Goldbach's weak conjecture*. Retrieved from Wikipédia: https://en.wikipedia.org/wiki/Goldbach%27s_weak_conjecture
- [3]- Helfgott., H. A. (2013). The ternary Goldbach conjecture is true. *arXiv*.
- [4]- Daniel, S. N. (2023). NUMERICAL VERIFICATION OF THE STRONG GOLDBACH CONJECTURE UP TO 9×10^18. *INT Journal of Mathematics*.
- [5]- KRAFFT, M. (1798). Essai sur les nombre premiers. *NOVA ACTA ACADEMIAE SCIENTIARVM IMPERIALIS PETROPOLITANAE TOMUS XV*, 217- 224. Retrieved from https://archive.org/details/novaactaacademia12impe/page/216/mode/2up
- [6]- Donato Saeli, M. S. (2012). La cometa di Goldbach e ... le altre. *arXiv*.
- [7]- Wang, X. (2022, October 27). A Concise Proof of Goldbach Conjecture. *RESEARCHGATE*. doi:DOI:10.20944/preprints202210.0420.v1
- [8]- Daniel Sankei, L. N. (2023). Partitioning an even number of the new formulation into all pairs of odd numbers. *Journal of Mathematical Problems, Equations and Statistics*. doi:https://doi.org/10.22271/math.2023.v4.i2a.104
- [9]- Härdig, J. (2020). Goldbach’s Conjecture.
- [10]- Shi H, M. Z. (2019). Even numbers are the sum of two prime numbers. *Greener Journal of Science, Engineering and Technological Research*. doi:DOI:10.15580/GJSETR.2019.1.040919068
- [11]- Greaves. (2001). *Sieves in Number Theory.* Springer.
- [12]- Simone, D. G. (2018). *Some important results*. Récupéré sur Let's prove Goldbach!: http://www.dimostriamogoldbach.it/en/?doing_wp_cron=1715406395.0186760425567626953125

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